

TWO-PARAMETER QUANTUM GROUPS AND DRINFEL'D DOUBLES

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ABSTRACT. We investigate two-parameter quantum groups corresponding to the general linear and special linear Lie algebras \mathfrak{gl}_n and \mathfrak{sl}_n . We show that these quantum groups can be realized as Drinfel'd doubles of certain Hopf subalgebras with respect to Hopf pairings. Using the Hopf pairing, we construct a corresponding R -matrix and a quantum Casimir element. We discuss isomorphisms among these quantum groups and connections with multiparameter quantum groups.

INTRODUCTION

In this work we study two two-parameter quantum groups $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ and $U = U_{r,s}(\mathfrak{sl}_n)$ corresponding to the Lie algebras \mathfrak{gl}_n and \mathfrak{sl}_n . Our Hopf algebra \tilde{U} is isomorphic as an algebra to Takeuchi's $U_{r,s^{-1}}$ (see [T]), but as a Hopf algebra, it has the opposite coproduct. (A different presentation of $U_{r,s^{-1}}$ was obtained by Kulish [K] (see also [Ji]).) As an algebra, \tilde{U} has generators $e_j, f_j, (1 \leq j < n)$, and $a_i^{\pm 1}, b_i^{\pm 1}$ ($1 \leq i \leq n$), and defining relations given in (R1)-(R7) below. The elements $e_j, f_j, \omega_j^{\pm 1}, (\omega'_j)^{\pm 1}$ ($1 \leq j < n$), where $\omega_j = a_j b_{j+1}$ and $\omega'_j = a_{j+1} b_j$, generate the subalgebra $U = U_{r,s}(\mathfrak{sl}_n)$.

We show that both \tilde{U} and U may be realized as Drinfel'd doubles of certain Hopf subalgebras with respect to suitable Hopf pairings. Using the Hopf pairing, we construct an R -matrix for \tilde{U} (which also works for U). For \tilde{U} -modules M and M' in category \mathcal{O} (defined in Section 4), there is an isomorphism $R_{M',M} : M' \otimes M \rightarrow M \otimes M'$. Moreover, the R -matrix satisfies the quantum Yang-Baxter equation and the hexagon identities. In [BW2], the R -matrix will be used to establish an analogue of Schur-Weyl duality in this setting: \tilde{U} has a natural n -dimensional module V , and the centralizer algebra $\text{End}_{\tilde{U}}(V^{\otimes k})$ is generated by a certain Hecke algebra $H_k(r, s)$. We construct a quantum Casimir element, which will play an essential role in [BW2] in proving that finite-dimensional modules in category \mathcal{O} are completely reducible.

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Jing's work [Ji], which treats the special case of \mathfrak{gl}_2 , adopts exactly the opposite approach to the one of this paper – it derives an analogue of the algebra \tilde{U} from one particular solution R of the quantum Yang-Baxter equation. Similarly, Chin and Musson [ChM] and Dobrev and Parashar [DP] study multiparameter quantum universal enveloping algebras defined as duals of quantum function algebras arising from R -matrices. In Section 6, we relate the two-parameter quantum groups considered here with certain special cases of these multiparameter quantum groups. Moreover, we determine conditions for isomorphisms among the two-parameter quantum groups. In particular, the standard one-parameter quantum group $U_q(\mathfrak{sl}_2)$ of [Ja] is isomorphic to a quotient of $U_{r,s}(\mathfrak{sl}_2)$ by the ideal generated by $\omega'_1 - \omega_1^{-1}$ whenever q is a square root of rs^{-1} . However, for $n \geq 3$, no such isomorphism exists (see Proposition 6.1).

Our motivation to study these two-parameter quantum groups came from our work [BW1] on down-up algebras. Down-up algebras were introduced in [BR] as a generalization of the algebra generated by the down and up operators on posets. They are unital associative algebras $A(\alpha, \beta, \gamma)$ over a field \mathbb{K} having generators d, u which satisfy the defining relations

$$\begin{aligned} d^2u &= \alpha dud + \beta ud^2 + \gamma d \\ du^2 &= \alpha udu + \beta u^2d + \gamma u, \end{aligned}$$

where α, β, γ are fixed but arbitrary scalars in \mathbb{K} . If $\gamma \neq 0$, then the down-up algebra $A(\alpha, \beta, \gamma)$ is isomorphic to $A(\alpha, \beta, 1)$. Thus, there are basically two different cases: $\gamma = 0$ and $\gamma = 1$. Examples of down-up algebras include the universal enveloping algebras of \mathfrak{sl}_2 , of the Heisenberg Lie algebra, and of the Lie superalgebra $\mathfrak{osp}(1, 2)$, which are $A(2, -1, 1)$, $A(2, -1, 0)$, and $A(0, -1, 1)$, respectively, and many of Witten's deformations of $U(\mathfrak{sl}_2)$ (see [B]). Down-up algebras exhibit many striking features including a Poincaré-Birkhoff-Witt type basis and a well-behaved representation theory ([BR], [KMP], [CaM], [Jor], [KK1], [KK2], [Ku], [BL, Sec. 4]). They are Noetherian domains whenever $\beta \neq 0$.

Essential to the structure of $A(\alpha, \beta, \gamma)$ are the roots of the equation

$$0 = t^2 - \alpha t - \beta = (t - r)(t - s),$$

Thus, $\alpha = r + s$ and $\beta = -rs$. When $rs \neq 0$ and $\gamma = 0$, the down-up algebra $A(\alpha, \beta, 0) = A(r + s, -rs, 0)$ can be extended by automorphisms to give a Hopf algebra $B(r + s, -rs, 0)$ (see [BW1]). This Hopf algebra is isomorphic to a subalgebra of $U_{r,s}(\mathfrak{sl}_3)$ when r and s are not roots of unity and to a quotient of a subalgebra when they are. It seemed natural to expect that there is a Drinfel'd double (quantum double) of the subalgebra, which yields a quantum group that depends on the two parameters r and s . In fact, that quantum group is $U_{r,s}(\mathfrak{sl}_3)$. That result is a very special case of our theorem showing that U and \tilde{U} are Drinfel'd doubles.

Throughout we will be working over a field \mathbb{K} , which is required to be algebraically closed from Section 3 to the end of the paper.

§1. PRELIMINARIES

Assume Φ is a finite root system of type A_{n-1} with Π a base of simple roots. We regard Φ as a subset of a Euclidean space $E = \mathbb{R}^n$ with an inner product $\langle \cdot, \cdot \rangle$. We let $\epsilon_1, \dots, \epsilon_n$ denote an orthonormal basis of E , and suppose $\Pi = \{\alpha_j = \epsilon_j - \epsilon_{j+1} \mid j = 1, \dots, n-1\}$ and $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\}$.

Fix nonzero elements r, s in a field \mathbb{K} . Here we assume $r \neq s$.

Let $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ be the unital associative algebra over \mathbb{K} generated by elements $e_j, f_j, (1 \leq j < n)$, and $a_i^{\pm 1}, b_i^{\pm 1}$ ($1 \leq i \leq n$), which satisfy the following relations.

$$(R1) \quad \text{The } a_i^{\pm 1}, b_j^{\pm 1} \text{ all commute with one another and } a_i a_i^{-1} = b_j b_j^{-1} = 1,$$

$$(R2) \quad a_i e_j = r^{\langle \epsilon_i, \alpha_j \rangle} e_j a_i \quad \text{and} \quad a_i f_j = r^{-\langle \epsilon_i, \alpha_j \rangle} f_j a_i,$$

$$(R3) \quad b_i e_j = s^{\langle \epsilon_i, \alpha_j \rangle} e_j b_i \quad \text{and} \quad b_i f_j = s^{-\langle \epsilon_i, \alpha_j \rangle} f_j b_i,$$

$$(R4) \quad [e_i, f_j] = \frac{\delta_{i,j}}{r-s} (a_i b_{i+1} - a_{i+1} b_i),$$

$$(R5) \quad [e_i, e_j] = [f_i, f_j] = 0 \quad \text{if} \quad |i-j| > 1,$$

$$(R6) \quad e_i^2 e_{i+1} - (r+s) e_i e_{i+1} e_i + r s e_{i+1} e_i^2 = 0,$$

$$e_i e_{i+1}^2 - (r+s) e_{i+1} e_i e_{i+1} + r s e_{i+1}^2 e_i = 0,$$

$$(R7) \quad f_i^2 f_{i+1} - (r^{-1} + s^{-1}) f_i f_{i+1} f_i + r^{-1} s^{-1} f_{i+1} f_i^2 = 0,$$

$$f_i f_{i+1}^2 - (r^{-1} + s^{-1}) f_{i+1} f_i f_{i+1} + r^{-1} s^{-1} f_{i+1}^2 f_i = 0.$$

The relations in (R6) are just the two defining relations of the down-up algebra $A(r+s, -rs, 0)$, while those in (R7) are the defining relations of $A(r^{-1} + s^{-1}, -r^{-1}s^{-1}, 0)$. In fact these two down-up algebras are isomorphic via the map that takes d to u' and u to d' (assuming d', u' are the generators of the latter) (see [BR]).

We will be interested in the subalgebra $U = U_{r,s}(\mathfrak{sl}_n)$ of $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ generated by the elements e_j, f_j, ω_j , and ω'_j ($1 \leq j < n$), where

$$(1.1) \quad \omega_j = a_j b_{j+1} \quad \text{and} \quad \omega'_j = a_{j+1} b_j.$$

These elements satisfy (R5)-(R7) along with the following relations:

$$(R1') \quad \text{The } \omega_i^{\pm 1}, \omega'_j^{\pm 1} \text{ all commute with one another and } \omega_i \omega_i^{-1} = \omega'_j (\omega'_j)^{-1} = 1,$$

$$(R2') \quad \omega_i e_j = r^{\langle \epsilon_i, \alpha_j \rangle} s^{\langle \epsilon_{i+1}, \alpha_j \rangle} e_j \omega_i \quad \text{and} \quad \omega_i f_j = r^{-\langle \epsilon_i, \alpha_j \rangle} s^{-\langle \epsilon_{i+1}, \alpha_j \rangle} f_j \omega_i,$$

$$(R3') \quad \omega'_i e_j = r^{\langle \epsilon_{i+1}, \alpha_j \rangle} s^{\langle \epsilon_i, \alpha_j \rangle} e_j \omega'_i \quad \text{and} \quad \omega'_i f_j = r^{-\langle \epsilon_{i+1}, \alpha_j \rangle} s^{-\langle \epsilon_i, \alpha_j \rangle} f_j \omega'_i,$$

$$(R4') \quad [e_i, f_j] = \frac{\delta_{i,j}}{r-s} (\omega_i - \omega'_i).$$

When $r = q$ and $s = q^{-1}$, the algebra $U_{r,s}(\mathfrak{gl}_n)$ modulo the ideal generated by the elements $b_i - a_i^{-1}$, $1 \leq i \leq n$, is just the quantum general linear group $U_q(\mathfrak{gl}_n)$, and $U_{r,s}(\mathfrak{sl}_n)$ modulo the ideal generated by the elements $\omega'_j - \omega_j^{-1}$, $1 \leq j < n$, is $U_q(\mathfrak{sl}_n)$.

Let $Q = \mathbb{Z}\Phi$ denote the root lattice and set $Q^+ = \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0}\alpha_i$. Then for any $\zeta = \sum_{i=1}^{n-1} \zeta_i \alpha_i \in Q$, we adopt the shorthand

$$(1.2) \quad \omega_\zeta = \omega_1^{\zeta_1} \cdots \omega_{n-1}^{\zeta_{n-1}}, \quad \omega'_\zeta = (\omega'_1)^{\zeta_1} \cdots (\omega'_{n-1})^{\zeta_{n-1}}$$

The following lemma is straightforward to check.

Lemma 1.3. *Suppose that $\zeta = \sum_{i=1}^{n-1} \zeta_i \alpha_i \in Q$. Then*

$$\begin{aligned} w_\zeta e_i &= r^{-\langle \epsilon_{i+1}, \zeta \rangle} s^{-\langle \epsilon_i, \zeta \rangle} e_i w_\zeta & w_\zeta f_i &= r^{\langle \epsilon_{i+1}, \zeta \rangle} s^{\langle \epsilon_i, \zeta \rangle} f_i w_\zeta \\ w'_\zeta e_i &= r^{-\langle \epsilon_i, \zeta \rangle} s^{-\langle \epsilon_{i+1}, \zeta \rangle} e_i w'_\zeta & w'_\zeta f_i &= r^{\langle \epsilon_i, \zeta \rangle} s^{\langle \epsilon_{i+1}, \zeta \rangle} f_i w'_\zeta. \end{aligned}$$

The algebras \widetilde{U} and U are Hopf algebras, where the $a_i^{\pm 1}, b_i^{\pm 1}$ are group-like elements, and the remaining coproducts are determined by

$$(1.4) \quad \Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes \omega'_i.$$

This forces the counit and antipode maps to be

$$(1.5) \quad \begin{aligned} \varepsilon(a_i) &= \varepsilon(b_i) = 1, & S(a_i) &= a_i^{-1}, & S(b_i) &= b_i^{-1} \\ \varepsilon(e_i) &= \varepsilon(f_i) = 0, & S(e_i) &= -\omega_i^{-1} e_i, & S(f_i) &= -f_i (\omega'_i)^{-1}. \end{aligned}$$

§2. DRINFEL'D DOUBLES

A *Hopf pairing* of two Hopf algebras H and H' is a bilinear form on $H' \times H$ satisfying the following properties (see [DT] or [Jo, 3.2.1]):

$$(2.1) \text{(i)} \quad (1, h) = \varepsilon_H(h), \quad (h', 1) = \varepsilon_{H'}(h')$$

$$\text{(ii)} \quad (h', hk) = (\Delta_{H'}(h'), h \otimes k) = \sum (h'_{(1)}, h)(h'_{(2)}, k)$$

$$\text{(iii)} \quad (h'k', h) = (h' \otimes k', \Delta_H(h)) = \sum (h', h_{(1)})(k', h_{(2)})$$

for all $h, k \in H$ and $h', k' \in H'$, where ε_H and $\varepsilon_{H'}$ denote the counits of H and H' , respectively, and Δ_H and $\Delta_{H'}$ are their coproducts. It is a consequence of the defining properties that a Hopf pairing satisfies

$$(S_{H'}(h'), h) = (h', S_H(h))$$

for all $h \in H$ and $h' \in H'$, where S_H and $S_{H'}$ denote the respective antipodes of H and H' .

Assume \tilde{B} is the Hopf subalgebra of \tilde{U} generated by $e_j, \omega_j^{\pm 1}$ ($1 \leq j < n$), and $a_n^{\pm 1}$. Let $(\tilde{B}')^{\text{coop}}$ be the Hopf algebra having the *opposite coproduct* to the Hopf subalgebra of \tilde{U} generated by $f_j, (\omega'_j)^{\pm 1}$ ($1 \leq j < n$), and $b_n^{\pm 1}$. Similarly B is the Hopf subalgebra of U generated by $e_j, \omega_j^{\pm 1}$ ($1 \leq j < n$), and $(B')^{\text{coop}}$ is generated by $f_j, (\omega'_j)^{\pm 1}$ ($1 \leq j < n$).

Lemma 2.2. *There are Hopf pairings of \tilde{B} and \tilde{B}' , respectively of B and B' .*

Proof. We begin by defining a bilinear form for $\tilde{B}' \times \tilde{B}$ first on the generators:

$$(2.3) \quad \begin{aligned} (f_i, e_j) &= \frac{\delta_{i,j}}{s-r}, \\ (\omega'_i, \omega_j) &= r^{\langle \epsilon_j, \alpha_i \rangle} s^{\langle \epsilon_{j+1}, \alpha_i \rangle} = r^{-\langle \epsilon_{i+1}, \alpha_j \rangle} s^{-\langle \epsilon_i, \alpha_j \rangle}, \\ (b_n, a_n) &= 1, \quad (b_n, \omega_j) = s^{-\langle \epsilon_n, \alpha_j \rangle}, \quad (\omega'_i, a_n) = r^{\langle \epsilon_n, \alpha_i \rangle}, \end{aligned}$$

for ($1 \leq i, j < n$). If ω'_i is replaced by $(\omega'_i)^{-1}$ in the second or third line of (2.3), we replace the image under the bilinear form by its inverse, and similarly for ω_j and ω_j^{-1} , a_n and a_n^{-1} , b_n and b_n^{-1} . On all other pairs of generators the form is 0.

In the second line of (2.3) we have applied the identity

$$(2.4) \quad \langle \epsilon_j, \alpha_i \rangle = -\langle \epsilon_{i+1}, \alpha_j \rangle,$$

which is quite useful in subsequent calculations.

The pairings in (2.3) may be *extended* to a bilinear form on $\tilde{B}' \times \tilde{B}$ by requiring that (2.1)(i)–(iii) hold. We need only verify that the relations in \tilde{B} and \tilde{B}' are preserved, ensuring that the bilinear form is well-defined. It will then be a Hopf pairing by definition. Restricting the form to $B' \times B$ gives the desired Hopf pairing of B and B' .

It is straightforward to check that the bilinear form preserves all the relations among the $\omega_i^{\pm 1}$, $a_n^{\pm 1}$ in \tilde{B} and the $(\omega'_j)^{\pm 1}$, $b_n^{\pm 1}$ in \tilde{B}' . We will verify that the form on $\tilde{B}' \times \tilde{B}$ preserves one of the remaining relations in \tilde{B} , and leave the other verifications to the reader. For each i , $1 \leq i < n$, consider

$$(X, e_i^2 e_{i+1} - (r+s)e_i e_{i+1} e_i + r s e_{i+1} e_i^2),$$

where X is any word in the generators of \tilde{B}' . By definition, this is equal to

$$(2.5) \quad (\Delta^2(X), e_i \otimes e_i \otimes e_{i+1} - (r+s)e_i \otimes e_{i+1} \otimes e_i + r s e_{i+1} \otimes e_i \otimes e_i).$$

In order for any one of these terms to be nonzero, X must involve exactly two f_i factors, one f_{i+1} factor, and arbitrarily many $(\omega'_j)^{\pm 1}$ and $b_n^{\pm 1}$ factors ($1 \leq j < n$). First assume that $X = f_i^2 f_{i+1}$. Then $\Delta^2(X)$ is equal to

$$(\omega'_i \otimes \omega'_i \otimes f_i + \omega'_i \otimes f_i \otimes 1 + f_i \otimes 1 \otimes 1)^2 (\omega'_{i+1} \otimes \omega'_{i+1} \otimes f_{i+1} + \omega'_{i+1} \otimes f_{i+1} \otimes 1 + f_{i+1} \otimes 1 \otimes 1).$$

The relevant terms of $\Delta^2(X)$ are

$$\begin{aligned} & f_i \omega'_i \omega'_{i+1} \otimes f_i \omega'_{i+1} \otimes f_{i+1} + \omega'_i f_i \omega'_{i+1} \otimes f_i \omega'_{i+1} \otimes f_{i+1} \\ & + f_i \omega'_i \omega'_{i+1} \otimes \omega'_i f_{i+1} \otimes f_i + \omega'_i f_i \omega'_{i+1} \otimes \omega'_i f_{i+1} \otimes f_i \\ & + (\omega'_i)^2 f_{i+1} \otimes f_i \omega'_i \otimes f_i + (\omega'_i)^2 f_{i+1} \otimes \omega'_i f_i \otimes f_i. \end{aligned}$$

Therefore (2.5) becomes

$$\begin{aligned} & (f_i \omega'_i \omega'_{i+1}, e_i)(f_i \omega'_{i+1}, e_i)(f_{i+1}, e_{i+1}) + (\omega'_i f_i \omega'_{i+1}, e_i)(f_i \omega'_{i+1}, e_i)(f_{i+1}, e_{i+1}) \\ & - (r+s)(f_i \omega'_i \omega'_{i+1}, e_i)(\omega'_i f_{i+1}, e_{i+1})(f_i, e_i) - (r+s)(\omega'_i f_i \omega'_{i+1}, e_i)(\omega'_i f_{i+1}, e_{i+1})(f_i, e_i) \\ & + rs((\omega'_i)^2 f_{i+1}, e_{i+1})(f_i \omega'_i, e_i)(f_i, e_i) + rs((\omega'_i)^2 f_{i+1}, e_{i+1})(\omega'_i f_i, e_i)(f_i, e_i) \\ & = \frac{1}{(s-r)^3} (1 + (\omega'_i, \omega_i) - (r+s)(\omega'_i, \omega_{i+1}) - (r+s)(\omega'_i, \omega_i)(\omega'_i, \omega_{i+1}) \\ & \quad + rs(\omega'_i, \omega_{i+1})^2 + rs(\omega'_i, \omega_{i+1})^2(\omega'_i, \omega_i)) \\ & = \frac{1}{(s-r)^3} (1 + rs^{-1} - (r+s)r^{-1} - (r+s)s^{-1} + r^{-1}s + 1) = 0. \end{aligned}$$

If $X = f_i f_{i+1} f_i$ or $X = f_{i+1} f_i^2$, then similar calculations show that (2.5) is equal to 0. Finally if X is *any* word involving exactly two f_i factors, one f_{i+1} factor, and arbitrarily many factors of $(\omega'_j)^{\pm 1}$ ($1 \leq j < n$) and $b_n^{\pm 1}$, then (2.5) will just be a scalar multiple of one of the quantities we have already calculated, and therefore will equal 0. (For example, if $X = f_i^2 \omega'_j f_{i+1}$, then (2.5) will be $(\omega'_j, \omega_{i+1})$ times the corresponding quantity for $X = f_i^2 f_{i+1}$.)

Analogous calculations show that the relations in \tilde{B}' are preserved. \square

As there is a Hopf pairing between \tilde{B} and \tilde{B}' , there is a skew-Hopf pairing between \tilde{B} and $(\tilde{B}')^{\text{coop}}$, where the latter is \tilde{B}' as an algebra, but with the opposite coproduct. Therefore, we may form the Drinfel'd double $D(\tilde{B}, (\tilde{B}')^{\text{coop}})$ as in [Jo, 3.2]. This is a Hopf algebra whose underlying coalgebra is $\tilde{B} \otimes (\tilde{B}')^{\text{coop}}$ (that is, $\tilde{B} \otimes (\tilde{B}')^{\text{coop}}$ as a vector space with the tensor product coalgebra structure). The algebra structure is given as follows: \tilde{B} and \tilde{B}' are identified as algebras with $\tilde{B} \otimes 1$ and $1 \otimes \tilde{B}'$ respectively in $D(\tilde{B}, (\tilde{B}')^{\text{coop}})$. Letting $a \in \tilde{B}$ and $b \in \tilde{B}'$, we have $(a \otimes 1)(1 \otimes b) = a \otimes b$ and

$$(1 \otimes b)(a \otimes 1) = \sum (S^{\text{coop}}(b_{(1)}), a_{(1)})(b_{(3)}, a_{(3)})a_{(2)} \otimes b_{(2)},$$

where S^{coop} denotes the antipode for $(\tilde{B}')^{\text{coop}}$. (This expression looks different from [Jo, Lemma 3.2.2 (iii)] as we have written our bilinear form in the reverse order.) A similar construction applies to B and B' .

Theorem 2.7. *$D(\tilde{B}, (\tilde{B}')^{\text{coop}})$ is isomorphic to \tilde{U} , and $D(B, (B')^{\text{coop}})$ is isomorphic to U .*

Proof. We will prove the first statement, and the second will follow by restricting to fewer generators. We will denote the image $e_i \otimes 1$ of e_i in $D(\tilde{B}, (\tilde{B}')^{\text{coop}})$ by \check{e}_i ,

and similarly for $\omega_i, a_n, f_i, \omega'_i, b_n$. Define a map $\varphi : D(\tilde{B}, (\tilde{B}')^{\text{coop}}) \rightarrow \tilde{U}$ by

$$\begin{aligned} \varphi(\check{e}_i) &= e_i, & \varphi(\check{f}_i) &= f_i, \\ \varphi(\check{\omega}_i^{\pm 1}) &= \omega_i^{\pm 1}, & \varphi((\check{\omega}'_i)^{\pm 1}) &= (\omega'_i)^{\pm 1}, & \varphi(\check{a}_n^{\pm 1}) &= a_n^{\pm 1}, & \varphi(\check{b}_n^{\pm 1}) &= b_n^{\pm 1}. \end{aligned}$$

First notice that by definition, φ preserves the coalgebra structures, the relations in \tilde{B} , and the relations in \tilde{B}' . Next we will verify that the mixed relations in $D(\tilde{B}, (\tilde{B}')^{\text{coop}})$ correspond to those in \tilde{U} .

To calculate $\check{f}_j \check{e}_i$, we use

$$\begin{aligned} \Delta^2(e_i) &= e_i \otimes 1 \otimes 1 + \omega_i \otimes e_i \otimes 1 + \omega_i \otimes \omega_i \otimes e_i, \\ (\Delta^{\text{coop}})^2(f_j) &= 1 \otimes 1 \otimes f_j + 1 \otimes f_j \otimes \omega'_j + f_j \otimes \omega'_j \otimes \omega'_j, \\ \text{and } S^{\text{coop}}(f_j) &= -f_j(\omega'_j)^{-1}, \end{aligned}$$

so that

$$\begin{aligned} \check{f}_j \check{e}_i &= (-f_j(\omega'_j)^{-1}, e_i)(\omega'_j, 1)\check{\omega}'_j + (1, \omega_i)(\omega'_j, 1)\check{e}_i \check{f}_j + (1, \omega_i)(f_j, e_i)\check{\omega}_i \\ &= -\frac{\delta_{i,j}}{s-r}\check{\omega}'_j + \check{e}_i \check{f}_j + \frac{\delta_{i,j}}{s-r}\check{\omega}_i. \end{aligned}$$

That is, $[\check{e}_i, \check{f}_j] = \delta_{i,j}(s-r)^{-1}(\check{\omega}_i - \check{\omega}'_i)$. Applying φ gives the desired relation in \tilde{U} .

We leave verification of the remaining relations to the reader. As \tilde{U} is generated by $e_i, f_i, \omega_i^{\pm 1}, (\omega'_i)^{\pm 1}$ ($1 \leq i < n$), a_n and b_n , the map φ is surjective, and there is an obvious inverse map. \square

§3. WEIGHT MODULES

Let $\Lambda = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$, which is the weight lattice of \mathfrak{gl}_n . Corresponding to any $\lambda \in \Lambda$ is an algebra homomorphism $\hat{\lambda}$ from the subalgebra \tilde{U}^0 of \tilde{U} generated by the elements $a_i^{\pm 1}, b_i^{\pm 1}$ ($1 \leq i \leq n$) to \mathbb{K} given by

$$(3.1) \quad \hat{\lambda}(a_i) = r^{\langle \epsilon_i, \lambda \rangle} \quad \text{and} \quad \hat{\lambda}(b_i) = s^{\langle \epsilon_i, \lambda \rangle}.$$

The restriction $\hat{\lambda} : U^0 \rightarrow \mathbb{K}$ to the subalgebra U^0 of U generated by $\omega_j^{\pm 1}, (\omega'_j)^{\pm 1}$ ($1 \leq j < n$) satisfies

$$(3.2) \quad \hat{\lambda}(\omega_j) = r^{\langle \epsilon_j, \lambda \rangle} s^{\langle \epsilon_{j+1}, \lambda \rangle} \quad \text{and} \quad \hat{\lambda}(\omega'_j) = r^{\langle \epsilon_{j+1}, \lambda \rangle} s^{\langle \epsilon_j, \lambda \rangle}.$$

Similarly for $U = U_{r,s}(\mathfrak{sl}_n)$, we let $\Lambda_{\mathfrak{sl}} = \mathbb{Z}\varpi_1 \oplus \cdots \oplus \mathbb{Z}\varpi_{n-1}$, the weight lattice of \mathfrak{sl}_n , where ϖ_i is the fundamental weight

$$\varpi_i = \epsilon_1 + \cdots + \epsilon_i - \frac{i}{n} \sum_{j=1}^n \epsilon_j.$$

If we fix n th roots $r^{1/n}$ and $s^{1/n}$ of r and s , respectively, then (3.2) can be used to define an algebra homomorphism $\hat{\lambda} : U^0 \rightarrow \mathbb{K}$ for any $\lambda \in \Lambda_{st}$.

Let M be a module for $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ of dimension $d < \infty$. If \mathbb{K} is algebraically closed (*which will be our assumption throughout the remainder of this work*), then

$$M = \bigoplus_{\chi} M_{\chi}$$

where each $\chi : \tilde{U}^0 \rightarrow \mathbb{K}$ is an algebra homomorphism, and M_{χ} is the generalized eigenspace given by

$$(3.3) \quad M_{\chi} = \{m \in M \mid (a_i - \chi(a_i)1)^d m = 0 = (b_i - \chi(b_i)1)^d m, \text{ for all } i\}.$$

When $M_{\chi} \neq 0$ we say that χ is a *weight* and M_{χ} is the corresponding *weight space*. (If M decomposes into genuine eigenspaces relative to \tilde{U}^0 (resp. U^0), then we say that \tilde{U}^0 (resp. U^0) *acts semisimply on M* .)

From relations (R2) and (R3) we deduce that

$$(3.4) \quad \begin{aligned} e_j M_{\chi} &\subseteq M_{\chi \cdot \widehat{\alpha}_j} \\ f_j M_{\chi} &\subseteq M_{\chi \cdot (-\widehat{\alpha}_j)} \end{aligned}$$

where $\widehat{\alpha}_j$ is as in (3.1), and $\chi \cdot \psi$ is the homomorphism with values $(\chi \cdot \psi)(a_i) = \chi(a_i)\psi(a_i)$ and $(\chi \cdot \psi)(b_i) = \chi(b_i)\psi(b_i)$. In fact, if $(a_i - \chi(a_i)1)^k m = 0$, then $(a_i - \chi(a_i)r^{(\epsilon_i, \alpha_j)}1)^k e_j m = 0$, and similarly for b_i and for f_j . This can be used to show that the sum of the eigenspaces is a submodule of M , and so if M is simple, this sum must be M itself. Thus, in (3.3), we may replace the power d by 1 whenever M is simple, and \tilde{U}^0 must act semisimply in this case. We also can see from (3.4) that for each simple M there is a homomorphism χ so that all the weights of M are of the form $\chi \cdot \hat{\zeta}$, where $\zeta \in Q$.

When all the weights of a module M are of the form $\hat{\lambda}$, where $\lambda \in \Lambda$, then for brevity we say that M has weights in Λ . Rather than writing $M_{\hat{\lambda}}$ for the weight space, we simplify the notation by writing M_{λ} .

Note then (3.4) can be rewritten as $e_j M_{\lambda} \subseteq M_{\lambda + \alpha_j}$ and $f_j M_{\lambda} \subseteq M_{\lambda - \alpha_j}$. Any simple \tilde{U} -module having one weight in Λ has all its weights in Λ .

We would like to argue that when rs^{-1} is not a root of unity, the elements e_j and f_j act nilpotently on any finite-dimensional module. For this we require the following result.

Proposition 3.5. *Suppose $\hat{\zeta} = \hat{\eta}$ as functions on \tilde{U}^0 (resp., U^0), where $\zeta, \eta \in \Lambda$ (resp., Λ_{st}). If rs^{-1} is not a root of unity, then $\zeta = \eta$.*

Proof. The result for \tilde{U}^0 is a straightforward consequence of (3.1), as rs^{-1} is not a root of 1. We will prove the statement for U^0 . We may assume $\zeta = \sum_{i=1}^{n-1} \zeta_i \alpha_i$ and $\eta = \sum_{i=1}^{n-1} \eta_i \alpha_i$, where $n\zeta_i, n\eta_i \in \mathbb{Z}$. The condition that $\hat{\zeta} = \hat{\eta}$ gives the equations

$$\begin{aligned}
\hat{\zeta}(\omega_i) &= r^{(\epsilon_i, \zeta)} s^{(\epsilon_{i+1}, \zeta)} = r^{\zeta_i - \zeta_{i-1}} s^{\zeta_{i+1} - \zeta_i} \\
&= \hat{\eta}(\omega_i) = r^{\eta_i - \eta_{i-1}} s^{\eta_{i+1} - \eta_i} \\
\hat{\zeta}(\omega'_i) &= r^{(\epsilon_{i+1}, \zeta)} s^{(\epsilon_i, \zeta)} = r^{\zeta_{i+1} - \zeta_i} s^{\zeta_i - \zeta_{i-1}} \\
&= \hat{\eta}(\omega'_i) = r^{\eta_{i+1} - \eta_i} s^{\eta_i - \eta_{i-1}},
\end{aligned}$$

where $\zeta_0 = \zeta_n = 0 = \eta_0 = \eta_n$. Letting $\mu_i = \zeta_i - \eta_i$, we may rewrite the above equations as

$$(3.6) \quad r^{\mu_i - \mu_{i-1}} s^{\mu_{i+1} - \mu_i} = 1$$

$$(3.7) \quad r^{\mu_{i+1} - \mu_i} s^{\mu_i - \mu_{i-1}} = 1.$$

Combining these we have

$$(3.8) \quad r^{\mu_{i+2} - \mu_{i+1} - \mu_i + \mu_{i-1}} = 1$$

$$(3.9) \quad s^{\mu_{i+2} - \mu_{i+1} - \mu_i + \mu_{i-1}} = 1$$

for $i = 1, \dots, n-2$. Since we are assuming that rs^{-1} is not a root of unity, not both r and s are roots of unity, so from these relations we see that

$$(3.10) \quad \mu_{i+2} - \mu_{i+1} - \mu_i + \mu_{i-1} = 0$$

for $i = 1, \dots, n-2$. We claim that the solution to the system of equations given by (3.10) satisfies

$$(3.11) \quad \mu_{2k} = k\mu_2 \quad \text{and} \quad \mu_{2k+1} = k\mu_2 + \mu_1.$$

This is true for μ_3 as $\mu_0 = 0$. Moreover, $\mu_4 = \mu_3 + \mu_2 - \mu_1 = 2\mu_2$. An easy induction proves the rest. Now $\mu_n = 0$, and using that fact in (3.10) we have

$$(3.12) \quad \mu_{n-1} = -\mu_{n-2} + \mu_{n-3}.$$

If $n = 2m$ for some m , then (3.11) and (3.12) give $\mu_2 = 0$. From (3.6), we have $(rs^{-1})^{\mu_1} = 1$, and because rs^{-1} is not a root of unity, this says $\mu_1 = 0$. The relations in (3.11) then show $\mu_i = \zeta_i - \eta_i = 0$ for all i . Hence $\zeta = \eta$ when n is even.

Now if instead $n = 2m+1$, then (3.11) and (3.12) show that $\mu_1 = -m\mu_2$. The equations in (3.6) and (3.7) imply

$$(3.13) \quad r^{-m\mu_2} s^{(m+1)\mu_2} = 1 \quad \text{and} \quad r^{(m+1)\mu_2} s^{-m\mu_2} = 1,$$

and hence that $(rs)^{\mu_2} = 1$. Then from (3.13) we see that $s^{(2m+1)\mu_2} = 1 = r^{(2m+1)\mu_2}$. As not both r and s are roots of unity, $\mu_2 = 0$. From this, the desired conclusion $\zeta = \eta$ follows. \square

Corollary 3.14. *Let M be a finite-dimensional module for $U_{r,s}(\mathfrak{sl}_n)$ or for $U_{r,s}(\mathfrak{gl}_n)$. If rs^{-1} is not a root of unity, then the elements e_i, f_i ($1 \leq i < n$) act nilpotently on M .*

Proof. Because M is a direct sum of its weight spaces, it suffices to argue that e_i and f_i act nilpotently on each M_χ . As the weights $\widehat{k\alpha_i}$ for $k = 1, 2, \dots$ are distinct by Proposition 3.5, and $e_i^k M_\chi \subseteq M_{\chi \cdot \widehat{k\alpha_i}}$, it must be that some power of e_i maps M_χ to 0. A similar argument applies to show that f_i is nilpotent also. \square

§4. R -MATRIX AND QUANTUM CASIMIR OPERATOR

Let \mathcal{O} denote the category of modules M for $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ which satisfy the conditions:

- (O1) \tilde{U}^0 acts semisimply on M , and the set $\text{wt}(M)$ of weights of M belongs to Λ :
 $M = \bigoplus_{\lambda \in \text{wt}(M)} M_\lambda$, where $M_\lambda = \{m \in M \mid a_i \cdot m = r^{\langle \epsilon_i, \lambda \rangle}, \quad b_i \cdot m = s^{\langle \epsilon_i, \lambda \rangle}$
for all $i\}$;
- (O2) $\dim_{\mathbb{K}} M_\lambda < \infty$ for all $\lambda \in \text{wt}(M)$;
- (O3) $\text{wt}(M) \subseteq \bigcup_{\mu \in F} (\mu - Q^+)$ for some finite set $F \subset \Lambda$.

The morphisms in \mathcal{O} are \tilde{U} -module homomorphisms. In defining category \mathcal{O} for the algebra $U = U_{r,s}(\mathfrak{sl}_n)$, we replace Λ by the weight lattice $\Lambda_{\mathfrak{sl}}$ of \mathfrak{sl}_n .

Category \mathcal{O} is closed under tensor product. For any two modules M and M' in \mathcal{O} , we construct a \tilde{U} -module isomorphism $R_{M',M} : M' \otimes M \rightarrow M \otimes M'$, by the method used by Jantzen [Ja, Chap. 7] for the quantum groups $U_q(\mathfrak{g})$. These isomorphisms work equally well for U -modules.

The map $R_{M',M}$ is the composite of three linear transformations P, \tilde{f}, Θ , which we now describe:

- (i) $P = P_{M',M} : M' \otimes M \rightarrow M \otimes M'$, $P(m' \otimes m) = m \otimes m'$.
- (ii) $\tilde{f} = \tilde{f}_{M,M'} : M \otimes M' \rightarrow M' \otimes M$ is such that $\tilde{f}(m \otimes m') = f(\lambda, \mu)(m \otimes m')$ when $m \in M_\lambda$ and $m' \in M'_\mu$, where the map $f : \Lambda \times \Lambda \rightarrow \mathbb{K}^\#$ is defined as follows. (The map \tilde{f} will be well-defined on modules in category \mathcal{O} by Proposition 3.5.)

Suppose that $\alpha_n = \epsilon_n$ so that $\alpha_i + \alpha_{i+1} + \cdots + \alpha_n = \epsilon_i$ for $i = 1, \dots, n$. Let $\Lambda = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$, (the weight lattice of \mathfrak{gl}_n). If $\lambda = \sum_{i=1}^n \lambda_i \alpha_i$ is in the weight lattice Λ , we define

$$(4.1) \quad \begin{aligned} \omega_\lambda &= \omega_1^{\lambda_1} \cdots \omega_{n-1}^{\lambda_{n-1}} a_n^{\lambda_n} \\ \omega'_\lambda &= (\omega'_1)^{\lambda_1} \cdots (\omega'_{n-1})^{\lambda_{n-1}} b_n^{\lambda_n}, \end{aligned}$$

which agrees with (1.2) in case $\lambda \in Q$. If also $\mu = \sum_{i=1}^n \mu_i \alpha_i$ is in Λ , we define

$$(4.2) \quad f(\lambda, \mu) = (\omega'_\mu, \omega_\lambda)^{-1}.$$

The values of this bilinear form are given by (2.1) and (2.3). It may be checked that for all $\lambda, \mu, \nu \in \Lambda$ and $1 \leq i, j < n$, the following hold:

$$(4.3) \quad \begin{aligned} f(\lambda + \mu, \nu) &= f(\lambda, \nu) f(\mu, \nu) \\ f(\lambda, \mu + \nu) &= f(\lambda, \mu) f(\lambda, \nu) \\ f(\alpha_j, \mu) &= r^{-\langle \epsilon_j, \mu \rangle} s^{-\langle \epsilon_{j+1}, \mu \rangle} \\ f(\lambda, \alpha_i) &= r^{\langle \epsilon_{i+1}, \lambda \rangle} s^{\langle \epsilon_i, \lambda \rangle}. \end{aligned}$$

If $\lambda, \mu \in Q$, then $\lambda_n = \mu_n = 0$ and

$$f(\lambda, \mu) = \prod_{i=1}^{n-1} (\hat{\lambda}(\omega'_i))^{\mu_i} = \prod_{i,j=1}^{n-1} (r^{\langle \epsilon_{i+1}, \alpha_j \rangle} s^{\langle \epsilon_i, \alpha_j \rangle})^{\mu_i \lambda_j} = \prod_{j=1}^{n-1} (\hat{\mu}(\omega_j))^{-\lambda_j}.$$

For $U = U_{r,s}(\mathfrak{sl}_n)$, where Λ is replaced by $\Lambda_{\mathfrak{sl}} = (1/n)Q$, this may be taken as the definition of f once we have fixed roots r^{1/n^2} and s^{1/n^2} of r and s in \mathbb{K} .

We will need to compute

$$f(\epsilon_i, \epsilon_j) = f(\alpha_i + \cdots + \alpha_n, \alpha_j + \cdots + \alpha_n).$$

Supposing first that $1 \leq i, j < n$, by (4.3) we have

$$\begin{aligned} f(\epsilon_i, \epsilon_j) &= (\omega'_{\alpha_j + \cdots + \alpha_n}, a_n)^{-1} \prod_{k=i}^{n-1} f(\alpha_k, \alpha_j + \cdots + \alpha_n) \\ &= r \prod_{k=i}^{n-1} r^{-\langle \epsilon_k, \alpha_j + \cdots + \alpha_n \rangle} s^{-\langle \epsilon_{k+1}, \alpha_j + \cdots + \alpha_n \rangle} \\ &= r \prod_{k=i}^{n-1} r^{-\langle \epsilon_k, \epsilon_j \rangle} s^{-\langle \epsilon_{k+1}, \epsilon_j \rangle} \\ &= \begin{cases} s^{-1} & \text{if } i < j \\ 1 & \text{if } i = j \\ r & \text{if } i > j. \end{cases} \end{aligned}$$

Now if $1 \leq i < n$, then

$$\begin{aligned} f(\epsilon_i, \epsilon_n) &= f(\alpha_i + \cdots + \alpha_n, \alpha_n) = (b_n, \omega_i)^{-1} \cdots (b_n, \omega_{n-1})^{-1} (b_n, a_n)^{-1} = s^{-1} \\ f(\epsilon_n, \epsilon_i) &= f(\alpha_n, \alpha_i + \cdots + \alpha_n) = (\omega_i, a_n)^{-1} \cdots (\omega_{n-1}, a_n)^{-1} (b_n, a_n)^{-1} = r \quad \text{and} \\ f(\epsilon_n, \epsilon_n) &= (b_n, a_n) = 1. \end{aligned}$$

As a result, the following holds:

Lemma 4.4. *For all $1 \leq i, j \leq n$, we have*

$$f(\epsilon_i, \epsilon_j) = \begin{cases} s^{-1} & \text{if } i < j \\ 1 & \text{if } i = j \\ r & \text{if } i > j. \end{cases}$$

(iii) Now we turn our attention to the construction of our final mapping. Observe it is a consequence of (R2) and (R3) that the subalgebra U^+ of \tilde{U} (or of $U = U_{r,s}(\mathfrak{sl}_n)$) generated by 1 and e_i ($1 \leq i < n$) has the following decomposition

$$U^+ = \bigoplus_{\zeta \in Q^+} U_{\zeta}^+$$

where

$$U_\zeta^+ = \{z \in U^+ \mid a_i z = r^{\langle \epsilon_i, \zeta \rangle} z a_i, \quad b_i z = s^{\langle \epsilon_i, \zeta \rangle} z b_i, \quad (1 \leq i < n)\}.$$

The weight space U_ζ^+ is spanned by all the monomials $e_{i_1} \cdots e_{i_\ell}$ such that $\alpha_{i_1} + \cdots + \alpha_{i_\ell} = \zeta$.

Similarly, the subalgebra U^- generated by 1 and the f_i 's has a decomposition $U^- = \bigoplus_{\zeta \in Q^+} U_{-\zeta}^-$, and the spaces U_ζ^+ and $U_{-\zeta}^-$ are nondegenerately paired.

Since $\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i$, we have

$$\Delta(x) \in \sum_{0 \leq \nu \leq \zeta} U_{\zeta - \nu}^+ \omega_\nu \otimes U_\nu^+$$

for all $x \in U_\zeta^+$. (In writing this, we are using the standard partial order on Q in which $\nu \leq \zeta$ if $\zeta - \nu \in Q^+$.) For each i , there are elements $p_i(x)$ and $p'_i(x) \in U_{\zeta - \alpha_i}^+$ such that

$$(4.5) \quad \begin{aligned} \Delta(x) &= x \otimes 1 + \sum_{i=1}^{n-1} p_i(x) \omega_i \otimes e_i + \text{the rest} \\ \Delta(x) &= \omega_\zeta \otimes x + \sum_{i=1}^{n-1} e_i \omega_{\zeta - \alpha_i} \otimes p'_i(x) + \text{the rest,} \end{aligned}$$

where in each case “the rest” refers to terms involving products of more than one e_j in the second factor (respectively, in the first factor). (Compare the expressions in Lemma 5.2 below.)

Lemma 4.6. (Compare [Ja, Lemma 6.14, 6.17].) For all $x \in U_\zeta^+$, $x' \in U_{\zeta'}^+$, and $y \in U^-$, the following hold:

- (i) $p_i(x x') = r^{\langle \epsilon_i, \zeta' \rangle} s^{\langle \epsilon_{i+1}, \zeta' \rangle} p_i(x) x' + x p_i(x')$.
- (ii) $p'_i(x x') = p'_i(x) x' + r^{-\langle \epsilon_{i+1}, \zeta \rangle} s^{-\langle \epsilon_i, \zeta \rangle} x p'_i(x)$.
- (iii) $(f_i y, x) = (f_i, e_i)(y, p'_i(x)) = (s - r)^{-1}(y, p'_i(x))$.
- (iv) $(y f_i, x) = (f_i, e_i)(y, p_i(x)) = (s - r)^{-1}(y, p_i(x))$.
- (v) $f_i x - x f_i = (s - r)^{-1}(p_i(x) \omega_i - \omega'_i p'_i(x))$.

Proof. The proofs of (i) and (ii) amount to equating the expressions for $\Delta(x x') = \Delta(x) \Delta(x')$. We demonstrate the second:

$$\begin{aligned}
\Delta(xx') &= \omega_{\zeta+\zeta'} \otimes xx' + \sum_{i=1}^{n-1} e_i \omega_{\zeta+\zeta'-\alpha_i} \otimes p'_i(xx') + \text{the rest} \\
\Delta(x)\Delta(x') &= \left(\omega_{\zeta} \otimes x + \sum_{i=1}^{n-1} e_i \omega_{\zeta-\alpha_i} \otimes p'_i(x) + \text{the rest} \right) \times \\
&\quad \left(\omega_{\zeta'} \otimes x' + \sum_{i=1}^{n-1} e_i \omega_{\zeta'-\alpha_i} \otimes p'_i(x') + \text{the rest} \right) \\
&= \omega_{\zeta+\zeta'} \otimes xx' \\
&\quad + \sum_{i=1}^{n-1} \left(e_i \omega_{\zeta+\zeta'-\alpha_i} \otimes p'_i(x)x' + \omega_{\zeta} e_i \omega_{\zeta'-\alpha_i} \otimes x p'_i(x') \right) + \text{the rest} \\
&= \omega_{\zeta+\zeta'} \otimes xx' \\
&\quad + \sum_{i=1}^{n-1} e_i \omega_{\zeta+\zeta'-\alpha_i} \otimes \left(p'_i(x)x' + r^{-\langle \epsilon_{i+1}, \zeta \rangle} s^{-\langle \epsilon_i, \zeta \rangle} x p'_i(x') \right) + \text{the rest},
\end{aligned}$$

(using Lemma 1.3). Equating terms gives (ii).

Now for (iii) we argue as follows using the second equation of (4.5):

$$\begin{aligned}
(f_i y, x) &= \sum (f_i, x_{(1)})(y, x_{(2)}) \\
&= (f_i, e_i \omega_{\zeta-\alpha_i})(y, p'_i(x)) \\
&= (1 \otimes f_i + f_i \otimes \omega'_i, \omega_{\zeta-\alpha_i} \otimes e_i)(y, p'_i(x)) \\
&= (f_i, e_i)(y, p'_i(x)).
\end{aligned}$$

Let's begin the proof of (v) by observing that it is true if $x = 1 \in U_0^+$ since $p_i(1) = p'_i(1) = 0$ for all i . The relation in (v) also holds for $x = e_j$, because $\Delta(e_j) = e_j \otimes 1 + w_j \otimes e_j$ implies that $p_i(e_j) = \delta_{i,j} = p'_i(e_j)$. We suppose the result is true for $x \in U_{\zeta}^+$ and $x' \in U_{\zeta'}^+$ and prove it for $xx' \in U_{\zeta+\zeta'}^+$. Now

$$\begin{aligned}
f_i xx' - xx' f_i &= (f_i x - x f_i)x' - x(f_i x' - x' f_i) \\
&= \frac{1}{s-r} \left((p_i(x)\omega_i - \omega'_i p'_i(x))x' + x(p_i(x')\omega_i - \omega'_i p'_i(x')) \right) \\
&= \frac{1}{s-r} \left(\left(r^{\langle \epsilon_i, \zeta' \rangle} s^{\langle \epsilon_{i+1}, \zeta' \rangle} p_i(x)x' + x(p_i(x')) \right) \omega_i \right. \\
&\quad \left. - \omega'_i \left(p'_i(x)x' + r^{-\langle \epsilon_{i+1}, \zeta \rangle} s^{-\langle \epsilon_i, \zeta \rangle} x p'_i(x') \right) \right) \\
&= \frac{1}{s-r} (p_i(xx')\omega_i - \omega'_i p'_i(xx')). \quad \square
\end{aligned}$$

Assuming for $y \in U_{-\zeta}^-$ that $p_i(y)$ and $p'_i(y)$ are defined by

$$(4.7) \quad \begin{aligned} \Delta(y) &= y \otimes \omega'_\zeta + \sum_{i=1}^n p_i(y) \otimes f_i \omega'_{\zeta - \alpha_i} + \text{the rest} \\ \Delta(y) &= 1 \otimes y + \sum_{i=1}^n f_i \otimes p'_i(y) \omega'_i + \text{the rest,} \end{aligned}$$

the same type of argument produces this analogue of Lemma 4.6:

Lemma 4.8. *For all $y \in U_{-\zeta}^-$, $y' \in U_{-\zeta'}^-$, and $x \in U^+$, the following hold:*

- (i) $p_i(y y') = p_i(y) y' + r^{\langle \epsilon_i, \zeta' \rangle} s^{\langle \epsilon_{i+1}, \zeta' \rangle} y p_i(y')$.
- (ii) $p'_i(y y') = r^{-\langle \epsilon_{i+1}, \zeta' \rangle} s^{-\langle \epsilon_i, \zeta' \rangle} p'_i(y) y' + y p'_i(y)$.
- (iii) $(y, e_i x) = (f_i, e_i)(p_i(y), x) = (s - r)^{-1}(p_i(y), (x))$.
- (iv) $(y, x e_i) = (f_i, e_i)(p'_i(y), x) = (s - r)^{-1}(p'_i(y), (x))$.
- (v) $e_i y - y e_i = (r - s)^{-1}(w_i p_i(y) - p'_i(y) \omega'_i)$.

Because the spaces U_ζ^+ and $U_{-\zeta}^-$ are nondegenerately paired, we may select a basis $\{u_k^\zeta\}_{k=1}^{d_\zeta}$, ($d_\zeta = \dim_{\mathbb{K}} U_\zeta^+$), for U_ζ^+ and a dual basis $\{v_k^\zeta\}_{k=1}^{d_\zeta}$ for $U_{-\zeta}^-$. Then for each $x \in U_\zeta^+$ and $y \in U_{-\zeta}^-$ we have

$$(4.9) \quad x = \sum_{k=1}^{d_\zeta} (v_k^\zeta, x) u_k^\zeta \quad \text{and} \quad y = \sum_{k=1}^{d_\zeta} (y, u_k^\zeta) v_k^\zeta.$$

For $\zeta \in Q^+ = \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \alpha_i$, we define

$$\Theta_\zeta = \sum_{k=1}^{d_\zeta} v_k^\zeta \otimes u_k^\zeta.$$

Set $\Theta_\zeta = 0$ if $\zeta \notin Q^+$.

- Lemma 4.10.** (i) $(a_i \otimes a_i) \Theta_\zeta = \Theta_\zeta (a_i \otimes a_i)$;
(ii) $(b_i \otimes b_i) \Theta_\zeta = \Theta_\zeta (b_i \otimes b_i)$;
(iii) $(e_i \otimes 1) \Theta_\zeta + (\omega_i \otimes e_i) \Theta_{\zeta - \alpha_i} = \Theta_\zeta (e_i \otimes 1) + \Theta_{\zeta - \alpha_i} (\omega'_i \otimes e_i)$;
(iv) $(1 \otimes f_i) \Theta_\zeta + (f_i \otimes \omega'_i) \Theta_{\zeta - \alpha_i} = \Theta_\zeta (1 \otimes f_i) + \Theta_{\zeta - \alpha_i} (f_i \otimes \omega_i)$;
for $1 \leq i < n$.

Proof. The first two are easy to check. We demonstrate (iv) and leave (iii) as an exercise. The calculation below will use (iii)-(v) of Lemma 4.6 and (4.9).

$$\begin{aligned}
& (1 \otimes f_i)\Theta_\zeta - \Theta_\zeta(1 \otimes f_i) \\
&= \sum_k v_k^\zeta \otimes (f_i u_k^\zeta - u_k^\zeta f_i) \\
&= \frac{1}{s-r} \sum_k v_k^\zeta \otimes \left(p_i(u_k^\zeta) \omega_i - \omega'_i p'_i(u_k^\zeta) \right) \\
&= \frac{1}{s-r} \sum_k v_k^\zeta \otimes \left(\sum_j (v_j^{\zeta-\alpha_i}, p_i(u_k^\zeta)) u_j^{\zeta-\alpha_j} \right) \omega_i \\
&\quad - \frac{1}{s-r} \sum_k v_k^\zeta \otimes \omega'_i \left(\sum_j (v_j^{\zeta-\alpha_i}, p'_i(u_k^\zeta)) u_j^{\zeta-\alpha_j} \right) \\
&= \sum_k v_k^\zeta \otimes \left(\sum_j (v_j^{\zeta-\alpha_i} f_i, u_k^\zeta) u_j^{\zeta-\alpha_i} \right) \omega_i \\
&\quad - \sum_k v_k^\zeta \otimes \omega'_i \left(\sum_j (f_i v_j^{\zeta-\alpha_i}, u_k^\zeta) u_j^{\zeta-\alpha_j} \right) \\
&= \sum_j \left(\sum_k (v_j^{\zeta-\alpha_i} f_i, u_k^\zeta) v_k^\zeta \right) \otimes u_j^{\zeta-\alpha_i} \omega_i \\
&\quad - \sum_j \left(\sum_k (f_i v_j^{\zeta-\alpha_i}, u_k^\zeta) v_k^\zeta \right) \otimes \omega'_i u_j^{\zeta-\alpha_j} \\
&= \sum_j v_j^{\zeta-\alpha_i} f_i \otimes u_j^{\zeta-\alpha_i} \omega_i - \sum_j f_i v_j^{\zeta-\alpha_i} \otimes \omega'_i u_j^{\zeta-\alpha_j} \\
&= \Theta_{\zeta-\alpha_i}(f_i \otimes \omega_i) - (f_i \otimes \omega'_i) \Theta_{\zeta-\alpha_i}.
\end{aligned}$$

Consequently, (iv) holds. \square

We now define

$$\Theta = \sum_{\zeta \in Q^+} \Theta_\zeta$$

(which we can think of as living in the completion of $\tilde{U} \otimes \tilde{U}$ where infinite sums are allowed). For fixed \tilde{U} -modules M and M' in \mathcal{O} , we may apply Θ to their tensor product:

$$\Theta = \Theta_{M, M'} : M \otimes M' \rightarrow M \otimes M'.$$

Note that $\Theta_\zeta : M_\lambda \otimes M'_\mu \rightarrow M_{\lambda-\zeta} \otimes M'_{\mu+\zeta}$ for all $\lambda, \mu \in \Lambda$, and because of condition (O3), there are only finitely many $\zeta \in Q^+$ such that $M'_{\mu+\zeta} \neq 0$. Hence, this is a well-defined linear transformation on $M \otimes M'$.

We can choose countable bases of weight vectors for both M and M' and their tensor products as a basis for $M \otimes M'$. Then ordering this basis appropriately shows that each Θ_ζ with $\zeta > 0$ has a strictly upper triangular matrix. Because $\Theta_0 = 1 \otimes 1$ acts as the identity transformation on $M \otimes M'$, $\Theta_{M,M'}$ is an invertible transformation.

Theorem 4.11. *Let M and M' be modules in \mathcal{O} . Then the map*

$$\Theta \circ \tilde{f} \circ P : M' \otimes M \rightarrow M \otimes M'$$

is an isomorphism of \tilde{U} -modules.

Proof. Since each of the maps is invertible, once we show that $\Theta \circ \tilde{f} \circ P$ is a \tilde{U} -module homomorphism, we will be done. The proof amounts to verifying that

$$(4.12) \quad \Delta(a)(\Theta \circ \tilde{f} \circ P)(m' \otimes m) = (\Theta \circ \tilde{f} \circ P)\Delta(a)(m' \otimes m)$$

holds for all $a \in \tilde{U}$, $m \in M_\lambda$ and $m' \in M'_\mu$. Because Δ is an algebra homomorphism, it suffices to check (4.12) on the generators e_i, f_i, a_i, b_i . We will present the computation just for $a = e_i$. In this case, the right side of (4.12) becomes

$$\begin{aligned} (\Theta \circ \tilde{f} \circ P)\Delta(e_i)(m' \otimes m) &= (\Theta \circ \tilde{f} \circ P)(e_i m' \otimes m + \omega_i m' \otimes e_i m) \\ &= (\Theta \circ \tilde{f})(m \otimes e_i m' + e_i m \otimes \omega_i m') \\ &= f(\lambda, \mu + \alpha_i)\Theta(m \otimes e_i m') + f(\lambda + \alpha_i, \mu)\Theta(e_i m \otimes \omega_i m') \\ &= f(\lambda, \mu + \alpha_i)\left(\sum_{\zeta} \Theta_{\zeta}\right)(1 \otimes e_i)(m \otimes m') \\ &\quad + f(\lambda + \alpha_i, \mu)\left(\sum_{\zeta} \Theta_{\zeta}\right)(e_i \otimes \omega_i)(m \otimes m') \end{aligned}$$

Now let's compute the left side using (iii) of Lemma 4.10:

$$\begin{aligned} \Delta(e_i)(\Theta \circ \tilde{f} \circ P)(m' \otimes m) &= f(\lambda, \mu)\Delta(e_i)\Theta(m \otimes m') \\ &= f(\lambda, \mu)(e_i \otimes 1)\left(\sum_{\zeta} \Theta_{\zeta}\right)(m \otimes m') \\ &\quad + f(\lambda, \mu)(\omega_i \otimes e_i)\left(\sum_{\zeta} \Theta_{\zeta - \alpha_i}\right)(m \otimes m') \\ &= f(\lambda, \mu)\left(\sum_{\zeta} \Theta_{\zeta}\right)(e_i m \otimes m') \\ &\quad + f(\lambda, \mu)\left(\sum_{\zeta} \Theta_{\zeta - \alpha_i}\right)(\omega_i m' \otimes e_i m') \\ &= f(\lambda, \mu)r^{-(\epsilon_i, \mu)}s^{-(\epsilon_{i+1}, \mu)}\left(\sum_{\zeta} \Theta_{\zeta}\right)(e_i m \otimes \omega_i m') \\ &\quad + f(\lambda, \mu)r^{(\epsilon_{i+1}, \lambda)}s^{(\epsilon_i, \lambda)}\left(\sum_{\zeta} \Theta_{\zeta - \alpha_i}\right)(m \otimes e_i m'). \end{aligned}$$

This expression can be seen to equal the previous one by (4.3). (Note in this computation we have made liberal use of the fact that $\sum_{\zeta} \Theta_{\zeta} = \sum_{\zeta} \Theta_{\zeta - \alpha_i}$ because of our convention that $\Theta_{\eta} = 0$ whenever $\eta \notin Q^+$.) \square

Quantum Casimir operator.

In this subsection, we construct a quantum Casimir operator which commutes with the action of \tilde{U} on any \tilde{U} -module in \mathcal{O} .

Again letting $\{u_k^{\zeta}\}_{k=1}^{d_{\zeta}}$ and $\{v_k^{\zeta}\}_{k=1}^{d_{\zeta}}$ be dual bases for U_{ζ}^+ and $U_{-\zeta}^-$ respectively ($d_{\zeta} = \dim_{\mathbb{K}} U_{\zeta}^+$), define

$$(4.15) \quad \Omega = \sum_{\zeta \in Q^+} \sum_{k=1}^{d_{\zeta}} S(v_k^{\zeta}) u_k^{\zeta},$$

where S denotes the antipode. Note that by (3.4), Ω preserves the weight spaces of any $M \in \mathcal{O}$.

Lemma 4.16. *For $M \in \mathcal{O}$, assume $m \in M_{\lambda}$. Then*

- (i) $\Omega e_i . m = (rs^{-1})^{-\langle \alpha_i, \lambda + \alpha_i \rangle} e_i \Omega . m,$
- (ii) $\Omega f_i . m = (rs^{-1})^{\langle \alpha_i, \lambda \rangle} f_i \Omega . m.$

Proof. Apply $m \circ (S \otimes 1)$ to Lemma 4.10 (iii), where m is the multiplication map. As S is an algebra anti-automorphism, this yields

$$\begin{aligned} & - \sum_{k=1}^{d_{\zeta}} S(v_k^{\zeta}) \omega_i^{-1} e_i u_k^{\zeta} + \sum_{k=1}^{d_{\zeta - \alpha_i}} S(v_k^{\zeta - \alpha_i}) \omega_i^{-1} e_i u_k^{\zeta - \alpha_i} \\ & = - \sum_{k=1}^{d_{\zeta}} \omega_i^{-1} e_i S(v_k^{\zeta}) u_k^{\zeta} + \sum_{k=1}^{d_{\zeta - \alpha_i}} (\omega_i')^{-1} S(v_k^{\zeta - \alpha_i}) u_k^{\zeta - \alpha_i} e_i. \end{aligned}$$

Now act on $m \in M_{\lambda}$ with the result, and sum over all $\zeta \in Q^+$. The two sums on the left side cancel, while the right side produces

$$\omega_i^{-1} e_i \Omega . m = (\omega_i')^{-1} \Omega e_i . m.$$

By definition, Ω preserves weight spaces, and so $e_i \Omega . m, \Omega e_i . m \in M_{\lambda + \alpha_i}$. Therefore we have

$$r^{-\langle \epsilon_i, \lambda + \alpha_i \rangle} s^{-\langle \epsilon_{i+1}, \lambda + \alpha_i \rangle} e_i \Omega . m = r^{-\langle \epsilon_{i+1}, \lambda + \alpha_i \rangle} s^{-\langle \epsilon_i, \lambda + \alpha_i \rangle} \Omega e_i . m,$$

which is equivalent to (i).

The proof of (ii) is virtually identical, and uses Lemma 4.10 (iv). \square

Now we introduce a certain function $g : \Lambda \rightarrow \mathbb{K}^{\#}$ on the weight lattice $\Lambda = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$ of \mathfrak{gl}_n . If ρ denotes the half sum of the positive

roots, then $2\rho = \sum_{j=1}^n (n+1-2j)\epsilon_j \in \Lambda$. We fix roots $r^{1/2}$ and $s^{1/2}$ and for $\lambda \in \Lambda$, set

$$(4.17) \quad g(\lambda) = (rs^{-1})^{\frac{1}{2}\langle \lambda + 2\rho, \lambda \rangle}.$$

Because $\langle \rho, \alpha_i \rangle = 1$ for all $i = 1, \dots, n-1$, it is straightforward to verify that

$$(4.18) \quad g(\lambda + \alpha_i) = (rs^{-1})^{\langle \alpha_i, \lambda + \alpha_i \rangle} g(\lambda)$$

for all $\lambda \in \Lambda$, $i \in \{1, \dots, n-1\}$.

For $M \in \mathcal{O}$, define the linear operator $\Xi : M \rightarrow M$ by

$$(4.19) \quad \Xi(m) = g(\lambda)m$$

for all $m \in M_\lambda$, $\lambda \in \Lambda$. Then Ξ is well-defined by Proposition 3.5. (It is necessary to first fix roots $r^{1/2n}$ and $s^{1/2n}$ of r and s in \mathbb{K} for $U = U_{r,s}(\mathfrak{sl}_n)$.)

Theorem 4.20. *The operator $\Omega\Xi : M \rightarrow M$ commutes with the action of \tilde{U} on any module $M \in \mathcal{O}$.*

Proof. As $\Omega\Xi$ preserves the weight spaces of M , it commutes with the action of a_i, b_i ($1 \leq i \leq n$). It remains to show that $\Omega\Xi$ commutes with e_i, f_i ($1 \leq i < n$). Let $m \in M_\lambda$. By Lemma 4.16 (i) and (4.18), we have

$$\begin{aligned} \Omega\Xi(e_i.m) &= g(\lambda + \alpha_i)\Omega e_i.m \\ &= (rs^{-1})^{\langle \alpha_i, \lambda + \alpha_i \rangle} g(\lambda)\Omega e_i.m \\ &= g(\lambda)e_i\Omega.m = e_i\Omega\Xi(m). \end{aligned}$$

The calculation for f_i is similar. \square

§5. YANG-BAXTER EQUATION AND HEXAGON IDENTITIES

For pairs M, M' of \tilde{U} -modules in category \mathcal{O} , we will show first that the maps $R_{M,M'} = \Theta \circ \tilde{f} \circ P : M \otimes M' \rightarrow M' \otimes M$ satisfy the quantum Yang-Baxter equation. That is, given three \tilde{U} -modules M, M', M'' in \mathcal{O} , we have $R_{12} \circ R_{23} \circ R_{12} = R_{23} \circ R_{12} \circ R_{23}$ as maps from $M \otimes M' \otimes M''$ to $M'' \otimes M' \otimes M$. This abbreviated notation is standard, for example R_{12} is an application of $R_{M,M'}$ to the first two of three factors and the identity map on the third factor.

We will need the following inner product relation. If $x \in U_\gamma^+$, $y \in U_{-\gamma}^-$, and $\zeta, \eta \in \mathcal{Q}$, then

$$(5.1) \quad (y\omega'_\zeta, x\omega_\eta) = (y, x)(\omega'_\zeta, \omega_\eta).$$

To derive this, we apply (4.5) and (4.7), keeping in mind that we need to take the *opposite* coproduct in the first position (equivalently, reverse the order of the factors in the second position):

$$\begin{aligned}
(y\omega'_\zeta, x\omega_\eta) &= (y \otimes \omega'_\zeta, \Delta(x)\Delta(\omega_\eta)) \\
&= (y, x\omega_\eta)(\omega'_\zeta, \omega_\eta) \\
&= (\Delta(y), \omega_\eta \otimes x)(\omega'_\zeta, \omega_\eta) \\
&= (y, x)(\omega'_\zeta, \omega_\eta).
\end{aligned}$$

Lemma 5.2. *Let $x \in U_\gamma^+$ and $y \in U_{-\gamma}^-$. Then*

- (i) $\Delta(x) = \sum_{0 \leq \zeta \leq \gamma} \sum_{i,j} (v_i^{\gamma-\zeta} v_j^\zeta, x) u_i^{\gamma-\zeta} \omega_\zeta \otimes u_j^\zeta$,
- (ii) $\Delta(y) = \sum_{0 \leq \zeta \leq \gamma} \sum_{i,j} (y, u_i^{\gamma-\zeta} u_j^\zeta) v_j^\zeta \otimes v_i^{\gamma-\zeta} \omega'_\zeta$.

Proof. As $x \in U_\gamma^+$, we have $\Delta(x) \in \sum_{0 \leq \zeta \leq \gamma} U_{\gamma-\zeta}^+ \omega_\zeta \otimes U_\zeta^+$. Let $c_{i,j}^\zeta \in \mathbb{K}$ be such that

$$\Delta(x) = \sum_{\zeta, i, j} c_{i,j}^\zeta u_i^{\gamma-\zeta} \omega_\zeta \otimes u_j^\zeta.$$

Then for all k, ℓ , and ν , we see from (5.1) that

$$\begin{aligned}
(v_k^{\gamma-\nu} v_\ell^\nu, x) &= (v_k^{\gamma-\nu} \otimes v_\ell^\nu, \Delta(x)) \\
&= \sum_{\zeta, i, j} c_{i,j}^\zeta (v_k^{\gamma-\nu}, u_i^{\gamma-\zeta} \omega_\zeta) (v_\ell^\nu, u_j^\zeta) = c_{k\ell}^\nu,
\end{aligned}$$

which proves (i). The argument for (ii) is similar. \square

Letting $\Theta^{\text{op}} = \sum_{\gamma \in Q^+} \sum_i u_i^\gamma \otimes v_i^\gamma$, $\Theta_{12} = \sum_{\gamma \in Q^+} \sum_i v_i^\gamma \otimes u_i^\gamma \otimes 1$, $\Theta_{ij}^f = \Theta_{ij} \circ \tilde{f}_{ij}$, and defining the other expressions in a like manner, we have the following identities for operators on $M \otimes M' \otimes M''$.

- Lemma 5.3.** (i) $(\Delta \otimes 1)(\Theta^{\text{op}}) \circ \tilde{f}_{31} \circ \tilde{f}_{32} = \Theta_{31}^f \circ \Theta_{32}^f$.
(ii) $\tilde{f}_{31} \circ \tilde{f}_{32} \circ \Theta_{12} = \Theta_{12} \circ \tilde{f}_{31} \circ \tilde{f}_{32}$.

Proof. Let $m \in M_\lambda$, $m' \in M'_\mu$, and $m'' \in M''_\nu$. Then by Lemma 5.2(i) and (4.9), the left side of (i) applied to $m \otimes m' \otimes m''$ is

$$\begin{aligned}
(\Delta \otimes 1)(\Theta^{\text{op}}) \circ \tilde{f}_{31} \circ \tilde{f}_{32} &= f(\nu, \mu) f(\nu, \lambda) (\Delta \otimes 1) \left(\sum_{\gamma, k} u_k^\gamma \otimes v_k^\gamma \right) \\
&= f(\nu, \mu) f(\nu, \lambda) \sum_{\gamma, k} \sum_{\zeta, i, j} (v_i^{\gamma-\zeta} v_j^\zeta, u_k^\gamma) u_i^{\gamma-\zeta} \omega_\zeta \otimes u_j^\zeta \otimes v_k^\gamma \\
&= f(\nu, \mu) f(\nu, \lambda) \sum_{\gamma, \zeta, i, j} u_i^{\gamma-\zeta} \omega_\zeta \otimes u_j^\zeta \otimes \left(\sum_k (v_i^{\gamma-\zeta} v_j^\zeta, u_k^\gamma) v_k^\gamma \right) \\
&= f(\nu, \mu) f(\nu, \lambda) \sum_{\gamma, \zeta, i, j} u_i^{\gamma-\zeta} \omega_\zeta \otimes u_j^\zeta \otimes v_i^{\gamma-\zeta} v_j^\zeta.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\Theta_{31}^f \circ \Theta_{32}^f(m \otimes m' \otimes m'') &= f(\nu, \mu) \sum_{\eta, \zeta, i, j} f(\nu - \zeta, \lambda) u_i^\eta m \otimes u_j^\zeta m' \otimes v_i^\eta v_j^\zeta m'' \\ &= f(\nu, \mu) f(\nu, \lambda) \sum_{\eta, \zeta, i, j} f(-\zeta, \lambda) u_i^\eta \otimes u_j^\zeta \otimes v_i^\eta v_j^\zeta (m \otimes m' \otimes m'').\end{aligned}$$

Changing variables in the first expression above to $\eta = \gamma - \zeta$, and noticing that $\omega_\zeta \cdot m = f(-\zeta, \lambda)m$, we obtain the second expression, proving (i).

Identity (ii) results from a simple calculation using (4.3). \square

We are now ready to verify the quantum Yang-Baxter equation.

Theorem 5.4. (Compare [Ja, §7.6].) $R_{12} \circ R_{23} \circ R_{12} = R_{23} \circ R_{12} \circ R_{23}$ as maps from $M \otimes M' \otimes M''$ to $M'' \otimes M' \otimes M$.

Proof. Note that $P_\sigma \circ \Theta_{ij}^f = \Theta_{\sigma(i)\sigma(j)}^f \circ P_\sigma$ for all permutations σ , and that the \tilde{f}_{ij} commute with one another. Applying Lemma 5.3 and Theorem 4.11, we have

$$\begin{aligned}R_{12} \circ R_{23} \circ R_{12} &= P_{12} \circ P_{23} \circ \Theta_{31}^f \circ \Theta_{32}^f \circ R_{12} \\ &= P_{12} \circ P_{23} \circ (\Delta \otimes 1)(\Theta^{\text{op}}) \circ \tilde{f}_{31} \circ \tilde{f}_{32} \circ \Theta_{12} \circ \tilde{f}_{12} \circ P_{12} \\ &= P_{12} \circ P_{23} \circ (\Delta \otimes 1)(\Theta^{\text{op}}) \circ \Theta_{12} \circ \tilde{f}_{31} \circ \tilde{f}_{32} \circ \tilde{f}_{12} \circ P_{12} \\ &= P_{12} \circ P_{23} \circ (\Delta \otimes 1)(\Theta^{\text{op}}) \circ \Theta_{12} \circ \tilde{f}_{12} \circ P_{12} \circ \tilde{f}_{32} \circ \tilde{f}_{31} \\ &= P_{12} \circ P_{23} \circ (\Delta \otimes 1)(\Theta^{\text{op}}) \circ R_{12} \circ \tilde{f}_{32} \circ \tilde{f}_{31} \\ &= P_{12} \circ P_{23} \circ R_{12} \circ (\Delta \otimes 1)(\Theta^{\text{op}}) \circ \tilde{f}_{32} \circ \tilde{f}_{31} \\ &= P_{12} \circ P_{23} \circ \Theta_{12}^f \circ P_{12} \circ \Theta_{31}^f \circ \Theta_{32}^f \\ &= \Theta_{23}^f \circ P_{12} \circ P_{23} \circ P_{12} \circ \Theta_{31}^f \circ \Theta_{32}^f \\ &= \Theta_{23}^f \circ P_{23} \circ P_{12} \circ P_{23} \circ \Theta_{31}^f \circ \Theta_{32}^f \\ &= \Theta_{23}^f \circ P_{23} \circ \Theta_{12}^f \circ P_{12} \circ \Theta_{23}^f \circ P_{23} \\ &= R_{23} \circ R_{12} \circ R_{23}. \quad \square\end{aligned}$$

Next we will verify the hexagon identities. For this we require two additional lemmas regarding operators on $M \otimes M' \otimes M''$.

Lemma 5.5. $(\Delta \otimes 1)(\Theta_\gamma) = \sum_{0 \leq \zeta \leq \gamma} (\Theta_{\gamma-\zeta})_{23} (\Theta_\zeta)_{13} (1 \otimes \omega'_\zeta \otimes 1)$, and $(1 \otimes \Delta)(\Theta_\gamma) = \sum_{0 \leq \zeta \leq \gamma} (\Theta_{\gamma-\zeta})_{12} (\Theta_\zeta)_{13} (1 \otimes \omega_\zeta \otimes 1)$.

Proof. By the definition of Θ_γ , Lemma 5.2(ii), and (4.9), we have

$$\begin{aligned}
(\Delta \otimes 1)(\Theta_\gamma) &= \sum_k \Delta(v_k^\gamma) \otimes u_k^\gamma \\
&= \sum_k \sum_{\zeta, i, j} (v_k^\gamma, u_i^{\gamma-\zeta} u_j^\zeta) v_j^\zeta \otimes v_i^{\gamma-\zeta} \omega'_\zeta \otimes u_k^\gamma \\
&= \sum_{\zeta, i, j} v_j^\zeta \otimes v_i^{\gamma-\zeta} \omega'_\zeta \otimes \left(\sum_k (v_k^\gamma, u_i^{\gamma-\zeta} u_j^\zeta) u_k^\gamma \right) \\
&= \sum_{\zeta, i, j} v_j^\zeta \otimes v_i^{\gamma-\zeta} \omega'_\zeta \otimes u_i^{\gamma-\zeta} u_j^\zeta \\
&= \sum_{0 \leq \zeta \leq \gamma} (\Theta_{\gamma-\zeta})_{23} (\Theta_\zeta)_{13} (1 \otimes \omega'_\zeta \otimes 1).
\end{aligned}$$

The second identity may be checked in just the same way. \square

Lemma 5.6. $\tilde{f}_{12} \circ (\Theta_\eta)_{13} = (\Theta_\eta)_{13} \circ (1 \otimes \omega_\eta \otimes 1) \circ \tilde{f}_{12}$, and $\tilde{f}_{23} \circ (\Theta_\eta)_{13} = (\Theta_\eta)_{13} \circ (1 \otimes \omega'_\eta \otimes 1) \circ \tilde{f}_{23}$.

Proof. Let $m \in M_\lambda$, $m' \in M'_\mu$, $m'' \in M''_\nu$. Then

$$\begin{aligned}
\tilde{f}_{12} \circ (\Theta_\eta)_{13}(m \otimes m' \otimes m'') &= f(\lambda - \eta, \mu) \sum_i v_i^\eta m \otimes m' \otimes u_i^\eta m'' \\
&= f(\lambda, \mu) f(-\eta, \mu) \sum_i v_i^\eta m \otimes m' \otimes u_i^\eta m'' \\
&= f(\lambda, \mu) \sum_i v_i^\eta m \otimes \omega_\eta m' \otimes u_i^\eta m'' \\
&= (\Theta_\eta)_{13} \circ (1 \otimes \omega_\eta \otimes 1) \circ \tilde{f}_{12}(m \otimes m' \otimes m'').
\end{aligned}$$

The second identity can be shown using $\omega'_\eta m' = f(\mu, \eta) m'$. \square

We continue with our assumption that M, M' , and M'' are \tilde{U} -modules in \mathcal{O} . To verify the hexagon identities, let \tilde{f}' denote the transformation on $M'' \otimes M \otimes M'$ taking $m'' \otimes m \otimes m' \in M''_\nu \otimes M_\lambda \otimes M'_\mu$ to $f(\nu, \lambda + \mu) m'' \otimes m \otimes m'$. Let \tilde{f}'' be the transformation on $M' \otimes M'' \otimes M$ taking $m' \otimes m'' \otimes m \in M'_\mu \otimes M''_\nu \otimes M_\lambda$ to $f(\mu + \nu, \lambda) m' \otimes m'' \otimes m$. As in [Ja, Thm. 3.18], the hexagon identities are equivalent to $R_{12} \circ R_{23} = (1 \otimes \Delta)(\Theta) \circ \tilde{f}' \circ P_{12} \circ P_{23}$ as maps from $M \otimes (M' \otimes M'') \rightarrow (M'' \otimes M) \otimes M'$, and $R_{23} \circ R_{12} = (\Delta \otimes 1)(\Theta) \circ \tilde{f}'' \circ P_{23} \circ P_{12}$ as maps from $(M \otimes M') \otimes M''$ to $M' \otimes (M'' \otimes M)$.

Theorem 5.7. *The hexagon identities hold, that is,*

- (i) $R_{12} \circ R_{23} = (1 \otimes \Delta)(\Theta) \circ \tilde{f}' \circ P_{12} \circ P_{23}$, and
- (ii) $R_{23} \circ R_{12} = (\Delta \otimes 1)(\Theta) \circ \tilde{f}'' \circ P_{23} \circ P_{12}$.

Proof. Let $m \otimes m' \otimes m'' \in M_\lambda \otimes M'_\mu \otimes M''_\nu$. By Lemma 5.5, the right side of (i) applied to $m \otimes m' \otimes m''$ gives

$$\begin{aligned} & (1 \otimes \Delta)(\Theta) \circ \tilde{f}' \circ P_{12} \circ P_{23}(m \otimes m' \otimes m'') \\ &= f(\nu, \lambda + \mu) \sum_{\gamma \in Q^+} \sum_{0 \leq \zeta \leq \gamma} (\Theta_{\gamma-\zeta})_{12} (\Theta_\zeta)_{13} (1 \otimes \omega_\zeta \otimes 1)(m'' \otimes m \otimes m'). \end{aligned}$$

On the other hand, by Lemma 5.6, the left side of (i) can be seen to equal

$$\begin{aligned} & \Theta_{12} \circ \tilde{f}_{12} \circ P_{12} \circ \Theta_{23} \circ \tilde{f}_{23} \circ P_{23}(m \otimes m' \otimes m'') \\ &= \Theta_{12} \circ \tilde{f}_{12} \circ \Theta_{13} \circ \tilde{f}_{13}(m'' \otimes m \otimes m') \\ &= \sum_{\eta \in Q^+} \Theta_{12} \circ (\Theta_\eta)_{13} \circ (1 \otimes \omega_\eta \otimes 1) \circ \tilde{f}_{12} \circ \tilde{f}_{13}(m'' \otimes m \otimes m') \\ &= f(\nu, \lambda) f(\nu, \mu) \sum_{\eta \in Q^+} \Theta_{12} \circ (\Theta_\eta)_{13} \circ (1 \otimes \omega_\eta \otimes 1)(m'' \otimes m \otimes m'). \end{aligned}$$

Then because $f(\nu, \lambda + \mu) = f(\nu, \lambda)f(\nu, \mu)$, a change of variables shows that this is equal to the right side of (i). The proof of (ii) is similar. \square

Remark 5.8. \mathcal{O} is a braided monoidal category with braiding $R = R_{M', M}$ for each pair of modules M', M in \mathcal{O} .

§6. ISOMORPHISMS AMONG QUANTUM GROUPS

We will now investigate isomorphisms among the two-parameter quantum groups, and their connections with multiparameter quantum groups. The case $n = 2$ is special, and the two parameters collapse to one in the following sense. Let $r, r', s, s' \in \mathbb{K}^\#$ and $r \neq s$, $r' \neq s'$. If $rs^{-1} = r'(s')^{-1}$, there is an isomorphism of Hopf algebras

$$\phi : U_{r,s}(\mathfrak{sl}_2) \rightarrow U_{r',s'}(\mathfrak{sl}_2)$$

given by $\phi(\omega^{\pm 1}) = \tilde{\omega}^{\pm 1}$, $\phi((\omega')^{\pm 1}) = (\tilde{\omega}')^{\pm 1}$, $\phi(e) = \tilde{e}$, $\phi(f) = r^{-1}r'\tilde{f}$, where “ $\tilde{}$ ” denotes generators of $U_{r',s'}(\mathfrak{sl}_2)$. (When dealing with \mathfrak{sl}_2 , we omit the subscript “1” on the generators.) The proof is a simple check that the relations and coproducts are preserved. In particular, if q is a square root of rs^{-1} , then $U_{r,s}(\mathfrak{sl}_2) \cong U_{q,q^{-1}}(\mathfrak{sl}_2)$ as Hopf algebras. Therefore the one-parameter quantum group $U_q(\mathfrak{sl}_2)$ is isomorphic to the quotient of $U_{r,s}(\mathfrak{sl}_2)$ by the ideal generated by $\omega' - \omega^{-1}$.

If $n \geq 3$, there is no such isomorphism, as the following proposition shows.

Proposition 6.1. *Let $n \geq 3$, and assume there is an isomorphism of Hopf algebras*

$$\phi : U_{r,s}(\mathfrak{sl}_n) \rightarrow U_{q,q^{-1}}(\mathfrak{sl}_n)$$

for some q . Then $r = q$ and $s = q^{-1}$.

Proof. Let ϕ be an isomorphism as hypothesized, and assume

$$\pi : U_{q,q^{-1}}(\mathfrak{sl}_n) \rightarrow U_q(\mathfrak{sl}_n)$$

is the surjection onto the standard one-parameter quantum group of [Ja] given by $\pi(e_i) = E_i$, $\pi(f_i) = F_i$, $\pi(\omega_i^{\pm 1}) = K_i^{\pm 1}$, $\pi((\omega'_i)^{\pm 1}) = K_i^{\mp 1}$. For $1 \leq i \leq n-1$, we have

$$(6.2) \quad \begin{aligned} \Delta(\pi\phi(e_i)) &= \pi\phi(\Delta(e_i)) = \pi\phi(e_i \otimes 1 + \omega_i \otimes e_i) \\ &= \pi\phi(e_i) \otimes 1 + \pi\phi(\omega_i) \otimes \pi\phi(e_i). \end{aligned}$$

Note that $\pi\phi(\omega_i)$ is necessarily a group-like element. Therefore $\pi\phi(e_i)$ is a skew-primitive element in $U_q(\mathfrak{sl}_n)$. By Theorem 5.4.1, Lemma 5.5.5, and the subsequent comments in [M], the set of group-like elements in $U_q(\mathfrak{sl}_n)$ is the group G generated by K_j ($1 \leq j \leq n-1$), and the skew-primitive elements together with the group-like elements span the subspace

$$\sum_{j=1}^{n-1} (\mathbb{K}E_j + \mathbb{K}F_j) + \mathbb{K}G.$$

Therefore

$$\pi\phi(e_i) = \sum_{j=1}^{n-1} \alpha_j^i E_j + \beta_j^i F_j + \sum_{g \in G} \gamma_g^i g$$

for some scalars $\alpha_j^i, \beta_j^i, \gamma_g^i$ ($1 \leq i, j \leq n-1$, $g \in G$). Consequently

$$\Delta(\pi\phi(e_i)) = \sum_{j=1}^{n-1} \alpha_j^i (E_j \otimes 1 + K_j \otimes E_j) + \beta_j^i (1 \otimes F_j + F_j \otimes K_j^{-1}) + \sum_{g \in G} \gamma_g^i g \otimes g.$$

By (6.2), this must be equal to

$$\sum_{j=1}^{n-1} (\alpha_j^i E_j \otimes 1 + \beta_j^i F_j \otimes 1 + \pi\phi(\omega_i) \otimes \alpha_j^i E_j + \pi\phi(\omega_i) \otimes \beta_j^i F_j) + \sum_g (\gamma_g^i g \otimes 1 + \pi\phi(\omega_i) \otimes \gamma_g^i g).$$

Comparing these two expressions, and noting that $\pi\phi(\omega_i) \neq 1$ as ϕ is an isomorphism (or by the comparison of expressions), we see first that

$$\sum_{j=1}^{n-1} \alpha_j^i E_j + \gamma_1^i 1 = \sum_{j=1}^{n-1} (\alpha_j^i E_j + \beta_j^i F_j) + \sum_{g \in G} \gamma_g^i g + \gamma_1^i \pi\phi(\omega_i).$$

Therefore $\gamma_g^i = 0$ for all g except $g \in \{1, \pi\phi(\omega_i)\}$, $\gamma_{\pi\phi(\omega_i)}^i = -\gamma_1^i$, and all $\beta_j^i = 0$. We now have

$$\pi\phi(e_i) = \sum_{1 \leq j \leq n-1} \alpha_j^i E_j + \gamma_1^i (1 - \pi\phi(\omega_i)).$$

A further comparison of coproducts yields

$$\begin{aligned} & \sum_{j=1}^{n-1} \alpha_j^i (E_j \otimes 1 + K_j \otimes E_j) + \gamma_1^i (1 \otimes 1 - \pi\phi(\omega_i) \otimes \pi\phi(\omega_i)) \\ &= \sum_{j=1}^{n-1} \alpha_j^i (E_j \otimes 1 + \pi\phi(\omega_i) \otimes E_j) \\ & \quad + \gamma_1^i (1 \otimes 1 - \pi\phi(\omega_i) \otimes 1) + \gamma_1^i (\pi\phi(\omega_i) \otimes 1 - \pi\phi(\omega_i) \otimes \pi\phi(\omega_i)), \end{aligned}$$

which implies that

$$\alpha_j^i(K_j - \pi\phi(\omega_i)) = 0, \quad \text{for all } 1 \leq j \leq n-1.$$

Thus all $\alpha_j^i = 0$ except possibly one, and some $\alpha_{j_i}^i$ must be nonzero as $\pi\phi$ is surjective. Therefore $\pi\phi(\omega_i) = K_{j_i}$.

Next we will apply $\pi\phi$ to relation (R2') and use the relations in $U_q(\mathfrak{sl}_n)$:

$$\begin{aligned} \pi\phi(\omega_i e_i) &= \pi\phi(rs^{-1}e_i\omega_i) \\ K_{j_i}(\alpha_{j_i}^i E_{j_i} + \gamma_1^i(1 - K_{j_i})) &= rs^{-1}(\alpha_{j_i}^i E_{j_i} + \gamma_1^i(1 - K_{j_i}))K_{j_i} \\ \alpha_{j_i}^i q^2 E_{j_i} K_{j_i} + \gamma_1^i K_{j_i} - \gamma_1^i K_{j_i}^2 &= \alpha_{j_i}^i rs^{-1} E_{j_i} K_{j_i} + \gamma_1^i rs^{-1} K_{j_i} - \gamma_1^i rs^{-1} K_{j_i}^2. \end{aligned}$$

This forces $rs^{-1} = q^2$, and $\gamma_1^i = 0$ as $r \neq s$. Applying relation (R2') again, we have

$$\begin{aligned} \pi\phi(\omega_i e_{i+1}) &= \pi\phi(se_{i+1}\omega_i) \\ K_{j_i} \alpha_{j_{i+1}}^{i+1} E_{j_{i+1}} &= \alpha_{j_{i+1}}^{i+1} s E_{j_{i+1}} K_{j_i} \\ \alpha_{j_{i+1}}^{i+1} q^{\langle \alpha_{j_i}, \alpha_{j_{i+1}} \rangle} E_{j_{i+1}} K_{j_i} &= \alpha_{j_{i+1}}^{i+1} s E_{j_{i+1}} K_{j_i}. \end{aligned}$$

Thus $s = q^{\langle \alpha_{j_i}, \alpha_{j_{i+1}} \rangle}$ must hold. By relations (R5) and (R6), we see that $|j_i - j_{i+1}| = 1$, so that in fact $s = q^{-1}$. Combining this with $rs^{-1} = q^2$ we now have $r = q$ as well. \square

Multiparameter quantum groups.

Deformations of GL_n involving $1 + \binom{n}{2}$ parameters were constructed independently by several authors (see [AST], [R], [S]). The dual version is a multiparameter universal enveloping algebra, which was studied by Chin and Musson [ChM] and Dobrev and Parashar [DP]. We will show that our two-parameter quantum groups are essentially special cases of these multiparameter quantum groups, as should be expected. We adopt the notation of Chin and Musson.

The $1 + \binom{n}{2}$ parameters are denoted λ and $p_{i,j}$ ($1 \leq i < j \leq n$) in [ChM]. Set $\lambda = rs^{-1}$ and $p_{i,j} = s^{-1}$ for all $i < j$. Let \widehat{U} be the Hopf algebra generated by E_i, F_i ($1 \leq i < n$) and $K_i^{\pm 1}, L_i^{\pm 1}$ ($1 \leq i \leq n$) with relations given by

$$(ChM0) \quad \text{The } K_i^{\pm 1}, L_j^{\pm 1} \text{ all commute with one another and } K_i K_i^{-1} = L_i L_i^{-1}.$$

$$(ChM1) \quad K_j E_i = r^{-\delta_{i,j}} s^{-\delta_{i,j-1}} E_i K_j \text{ and } K_j F_i = r^{\delta_{i,j}} s^{\delta_{i,j-1}} F_i K_j.$$

$$(ChM2) \quad L_j E_i = r^{\delta_{i,j-1}} s^{\delta_{i,j}} E_i L_j \text{ and } L_j F_i = r^{-\delta_{i,j-1}} s^{-\delta_{i,j}} F_i L_j.$$

$$(ChM3) \quad E_i F_i - r^{-1} s F_i E_i = (r^{-1} s - 1)(L_{i+1} K_{i+1} L_i^{-1} K_i^{-1} - 1).$$

$$(ChM4) \quad E_i F_j = r^{\delta_{i,j+1}} s^{-\delta_{i,j-1}} F_j E_i \text{ if } i \neq j.$$

$$(ChM5) \quad \text{ad}_\ell(E_i)^{1-\langle \alpha_i, \alpha_j \rangle}(E_j) = 0 \text{ and } \text{ad}_\ell(F_i)^{1-\langle \alpha_i, \alpha_j \rangle}(F_j) = 0 \text{ if } i \neq j.$$

These relations are given in [ChM, Thm. 4.8] (for more general $\lambda, p_{i,j}$) as relations for a Hopf algebra that is defined as a subalgebra of the finite dual A^0 of a multiparameter quantum function algebra A . In addition, Chin and Musson

give one more set of conditions: (ChM6) those relations among the K_i, L_j which determine the structure of the group they generate *as a subgroup of the group of units of A^0* . This results in a Hopf algebra \overline{U} (denoted U in their paper). Thus the multiparameter Hopf algebra \overline{U} of Chin and Musson is the quotient of our \widehat{U} by their relations (ChM6) (in case $\lambda = rs^{-1}$ and $p_{i,j} = s^{-1}$).

The Hopf structure of \widehat{U} is defined by requiring K_i, L_i to be group-like elements and

$$\Delta(E_i) = E_i \otimes 1 + L_{i+1}L_i^{-1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_{i+1}K_i^{-1} \otimes F_i.$$

Proposition 6.3. *There is a Hopf algebra morphism $\phi : \widehat{U} \rightarrow U_{r,s}(\mathfrak{gl}_n)$ given by*

$$\begin{aligned} \phi(L_i) &= a_1 \cdots a_{i-1} b_{i+1}^{-1} \cdots b_n^{-1}, \\ \phi(K_i) &= b_1^{-1} \cdots b_{i-1}^{-1} a_{i+1} \cdots a_n, \\ \phi(E_i) &= -s^{-1}(r-s)^2 e_i, \\ \phi(F_i) &= (\omega'_i)^{-1} f_i. \end{aligned}$$

The proof is just a check that the relations of \widehat{U} are preserved by ϕ , and that the coproducts of the generators correspond to the coproducts of their images.

Remark 6.4. Note that the kernel and cokernel of ϕ are generated by group-like elements. We have $\omega_i = \phi(L_i^{-1}L_{i+1})$ and $\omega'_i = \phi(K_iK_{i+1}^{-1})$, which implies that $U_{r,s}(\mathfrak{sl}_n)$ is contained in the image of ϕ . In case $n = 2$, it is easy to see that ϕ is an isomorphism. If $n = 3$, straightforward calculations show that the cokernel of ϕ is precisely $\mathbb{K}\langle a_1^{\pm 1} \rangle$.

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