Research Program

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I work on the cohomology, structure, and representations of various types of rings, such as Hopf algebras and group-graded algebras. I have a very active research program, including collaborations with many mathematicians, and work with postdocs and graduate students. I will briefly summarize some of my ongoing projects and future plans, which fall loosely into the three categories below.

Hochschild cohomology and deformations. A large part of my research program is currently focused on Hochschild cohomology and deformations of algebras. Hochschild cohomology is important in deformation theory, since a deformation of an algebra is infinitesimally a Hochschild 2-cocycle, and obstructions to lifting 2-cocycles to deformations live in degree 3 cohomology. My work in deformation theory began with [9], a paper I wrote with Căldăraru and Giaquinto. We were inspired by some examples of Vafa and Witten, which are deformations of certain skew group algebras arising from orbifolds. Specifically, these are the skew group algebras $S(V) \rtimes G$ (and twisted versions), where G is a finite group acting by graded automorphisms on the symmetric algebra $S(V)$ of a vector space V (i.e. a polynomial ring in a basis of V). Later I saw how to use techniques from [9] to yield a new approach to understanding graded Hecke algebras and symplectic reflection algebras (introduced independently in different settings by Drinfeld [11], Lusztig [24], and Etingof and Ginzburg [12]), for they are simply special types of deformations of $S(V) \rtimes G$. As first steps in this direction, I wrote [43], giving more general types of deformations (but under strong restrictions on G), and [44], generalizing results of Ram and Shepler [31] on graded Hecke algebras to a twisted case.

I further developed my ideas from [44] in work with Shepler [32], where we gave a large class of new examples of graded Hecke algebras corresponding to the complex reflection group $G(r, 1, n) \cong \mathbb{Z}/r\mathbb{Z} \wr S_n$, as a result of our explicit computations of cohomology for these groups. We continued our program to understand generally deformations of $S(V) \rtimes G$, motivated by the many interesting deformations already known, and by connections to geometry and combinatorics: Shepler and I recently have written three more papers [34, 33, 35] on various details of the algebraic structure of the Hochschild cohomology $\mathrm{HH}^*(S(V) \rtimes G)$ of $S(V) \rtimes G$. Its structure as a graded vector space was found independently by Ginzburg and Kaledin [21] and Farinati [15]. It has a cup product and a graded Lie bracket. The bracket is of interest in deformation theory, since a necessary condition for a Hochschild 2-cocycle to lift to a deformation is that its square bracket be a coboundary. In [32], Shepler and I give explicit chain maps from the bar resolution to the Koszul resolution of $S(V)$. Applying functors $\otimes$ or Hom with respect to various modules, we recover some maps in the literature and obtain some new maps. We use our chain maps for computation: The Hochschild cohomology $\mathrm{HH}^*(S(V) \rtimes G)$ is generally found using the Koszul complex, while the bracket is defined on the bar complex. Chain maps are needed to move back and forth between complexes for computational purposes. We use our maps in [34] to prove some general results on the ring structure of $\mathrm{HH}^*(S(V) \rtimes G)$. We find generators for this Hochschild cohomology ring, and identify orbifold cohomology as a subalgebra (under some conditions). The structure has a nice combinatorial description for some types of reflection groups. In [35], we use our chain maps to compute brackets in $\mathrm{HH}^*(S(V) \rtimes G)$, finding some sufficient conditions for brackets to be 0. We are currently working on more general deformations of $S(V) \rtimes G$, using our knowledge of Hochschild cohomology, and on analogous questions in positive characteristic (dividing the order of G) where the cohomology becomes more complicated due in part to the nonsemisimplicity of the group algebra.

This line of research has been very fruitful for my three current PhD students and for a former postdoc Naidu. Naidu and students Shakalli and Shroff have worked or are working on related problems. They held joint workshops with my collaborator Shepler's students to learn from each other. The workshops took place on October 23, 2010 at Texas A&M University and on April 23, 2011 at the University of North Texas.

One project of mine with Naidu and Shroff replaced $S(V)$ with a twisted polynomial ring $S_q(V)$ having a finite group action. Such actions have recently turned up in the literature, and so have deformations of $S_q(V) \rtimes G$ (see for example [2]). In [28], we computed the Hochschild cohomology of $S_q(V) \rtimes G$ in case G acts diagonally on a fixed basis. We continue to work on more general group actions as well as understanding corresponding deformations of $S_q(V) \rtimes G$ from this point of view. Shakalli is approaching such deformations from a different direction, generalizing some of the results in my paper [43].

Finite generation of cohomology. Another part of my research program involves a (graded commutative) cohomology ring $\mathrm{Ext}^*_H(k, k)$ of a finite dimensional Hopf algebra H over a field k. Historically one would take H to be a group algebra of a finite group. Finite group cohomology is a well-developed subject in which much is gained from the knowledge that cohomology is finitely generated: Affine varieties are defined to study modules and learn coarse information about them even when the representation type is wild. Many mathematicians have worked to extend this theory to other types of algebras. Friedlander and Suslin [19] proved finite generation of cohomology for cocommutative H, vastly generalizing earlier work of Venkov and Evens for finite group algebras, and Ginzburg and Kumar [22] proved finite generation of cohomology for small quantum groups. A conjecture of Etingof and Ostrik [13] implies that the cohomology of any finite dimensional Hopf algebra is finitely generated. A similar conjecture of Snashall and Solberg [36], modified after an example of Xu [45] appeared, regards Hochschild cohomology of finite dimensional algebras.

I am attacking this conjecture and its implications from many angles. My first work along these lines was with Mastnak, Pevtsova, and Schauenburg [25], in which we proved finite generation of the cohomology of any finite dimensional pointed Hopf algebra with abelian group of grouplike elements (under some hypotheses on the parameters); these Hopf algebras are a very general collection to which the small quantum groups belong. In our proof, we used the recent classification of such Hopf algebras by Andruskiewitsch and Schneider [1]. I plan to prove finite generation for other special classes of Hopf algebras and for Hopf algebras constructed from well-understood component algebras. My student Nguyen may choose to work on one aspect of this problem. As a related project, my student Shroff is working on questions of finite generation of cohomology for other types of algebras, particularly some quotients of PBW algebras that would generalize those in my paper [25].

Finite generation questions are of interest in their own right, but there are also important applications: When cohomology is finitely generated, one may define algebraic varieties associated to modules, called support varieties, that contain useful information. In work with Pevtsova [29], I studied varieties for modules of quantum elementary abelian groups. These are analogs, in the quantum group setting, of the elementary abelian groups that play such an important role in group cohomology (such as in Quillen's stratification). With Pevtsova, I defined a representation-theoretic variety, called a "rank variety" by analogy with finite group cohomology, and showed it was equivalent to the cohomologically-defined support variety. This equivalence allowed us to prove a tensor product property, namely that the variety of a tensor product of modules is the intersection of their varieties.

More recently, in work with Feldvoss [17], I further developed and applied the theory of varieties for modules of finite dimensional Hopf algebras (under the assumption of finite generation of cohomology). In particular we proved a tensor product property for a special class of modules. This property allowed us to adapt methods of Farnsteiner [16], developed for cocommutative Hopf algebras, to prove a conjecture of Cibils on the representation type of small quantum groups: If the rank of a simple Lie algebra $\mathfrak{g}$ is at least 2, then both the small quantum group $u_q(\mathfrak{g})$ and its Borel-type subalgebra $u_q(\mathfrak{b})$ are wild. In a second paper with Feldvoss [18] we considered support varieties defined via *Hochschild* cohomology to obtain results on representation type in more general settings.

Structure and representations of Hopf algebras. Some of my earliest research was on Hopf algebras, beginning with my thesis on representations of the quantum double (or Drinfeld double) of a finite group (published in [37]) and a generalization to the twisted case [38]. These are (quasi) Hopf algebras that first arose in connection to conformal field theory and vertex operator algebras. My familiarity with them has been extremely useful more recently, due to current interest in group-theoretical fusion categories. In work with Nikshych and postdoc Naidu, I gave in [27] a complete description of all fusion subcategories of the representation category of a twisted quantum double of a finite group. This is equivalent to a description of all group-theoretical braided fusion categories. In work with Etingof and Rowell [14], I proved that braid group representations arising from the twisted quantum double of a finite group always factor through a finite group, in stark contrast to those arising from quantum groups.

In a somewhat different direction, with Montgomery in [26] I had proven a special case of a conjecture of Kaplansky: If H is a Hopf algebra in characteristic 0 that is constructed by a sequence of crossed products of group algebras and their duals, then the dimension of a simple H-module divides the dimension of

H. At the time, all known semisimple Hopf algebras in characteristic 0 had that form. Some of my early independent work on representations of more general finite dimensional Hopf algebras is in [39, 40], the latter of which led me to develop a Clifford theory for algebras in [41], that is a connection between representations of an algebra and representations of a subalgebra (classical Clifford theory applies to groups).

Later I began to work on the representations and structure of quantum groups with Benkart, and we wrote several papers in this area [4, 5, 6, 7, 8], connecting the combinatorial down-up algebras with two-parameter quantum groups, studying their representations, and using them in deformation theory. In [3] with Benkart and my former PhD student Pereira, we found the precise effect of cocycle twisting on modules for a small quantum group. We used Radford's explicit construction of modules [30] and the computer algebra system SINGULAR::PLURAL. In a recent paper written with Cibils and former postdoc Lauve [10], I found a new series of finite dimensional pointed Hopf algebras, which are finite analogs of quantum Jordanian planes. These occur only in positive characteristic, and are built from non-semisimple representations of finite cyclic groups. To find and understand these Hopf algebras, we used Cibils' Hopf quiver method and a *Maple* program. Our motivation was the open question: Which pointed Hopf algebras are finite dimensional? It is easy to construct such Hopf algebras from finite group representations, but hard to decide when they are finite dimensional. This remains an open problem in which many Hopf algebraists are interested.

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