Math 131 Lecture Notes
Section 2.7 – The Derivative as a Function

The derivative of a function $f$ at a fixed number can be given as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

and can be interpreted geometrically as the slope of the tangent line to graph of $f$ at the point $(x, f(x))$. The domain of $f'$ is the set \{x | $f'(x)$ exists\} and may be smaller than the domain of $f$.

Example: The graph of a function $f$ is given below. Use it to sketch the graph of the derivative $f'$.

![Graph of $f(x)$]

Example: Let $P(t)$ be the population of a town during a boom and decline, given at time $t$. The table below gives midyear values of $P(t)$, in thousands, from 1990-2000. Construct a table of values for the derivative of this function.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$P(t)$</th>
<th>$P'(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1990</td>
<td>3.1</td>
<td></td>
</tr>
<tr>
<td>1991</td>
<td>5.3</td>
<td></td>
</tr>
<tr>
<td>1992</td>
<td>8.0</td>
<td></td>
</tr>
<tr>
<td>1993</td>
<td>10.2</td>
<td></td>
</tr>
<tr>
<td>1994</td>
<td>9.8</td>
<td></td>
</tr>
<tr>
<td>1995</td>
<td>7.7</td>
<td></td>
</tr>
<tr>
<td>1996</td>
<td>6.5</td>
<td></td>
</tr>
<tr>
<td>1997</td>
<td>5.7</td>
<td></td>
</tr>
<tr>
<td>1998</td>
<td>5.2</td>
<td></td>
</tr>
<tr>
<td>1999</td>
<td>4.8</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>4.7</td>
<td></td>
</tr>
</tbody>
</table>
Example: Find the derivative of $f(x) = x^2 - x$.

Example: Find the derivative of $f(x) = \sqrt[3]{x}$. State the domain of $f'$.

Example: Find $f'$ for $f(x) = \frac{x + 2}{1 - x}$.
Alternative notations for the derivative:

\[ f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} \]

The symbols \( D \) and \( \frac{d}{dx} \) are called **differentiation operators** because they indicate the operation of **differentiation** or calculating a derivative.

To indicate the value of a derivative at a specific number \( a \), we can use the notation

\[ \frac{dy}{dx}_{x=a} \quad \text{or} \quad \left. \frac{dy}{dx} \right|_{x=a} \]

A function \( f \) is **differentiable at \( a \)** if \( f'(a) \) exists. It is **differentiable on an open interval** \((a, b)\) [or \((a, \infty)\) or \((-\infty, a)\) or \((-\infty, \infty)\)] if it is differentiable at every number in the interval.

**Example:** Where is the function \( f(x) = |x| \) differentiable?

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**Theorem:** If \( f \) is differentiable at \( a \), then \( f \) is continuous at \( a \).

**Note:** The converse of the theorem is **not** true.

The graphs below illustrate three possibilities for functions that are not differentiable at \( a \).

(a) A corner  
(b) A discontinuity  
(c) A vertical tangent
If \( f \) is a differentiable function, then its derivative \( f' \) is also a function, so \( f' \) may have a derivative, denoted by \( f'' \) and called the **second derivative** of \( f \).

**Notation:** \[
\frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2y}{dx^2}
\]

**Example:** If \( f(x) = x^2 - x \), find and interpret \( f''(x) \).

If \( s = s(t) \) is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity of \( v(t) \) of the object as a function of time:

\[
v(t) = s'(t) = \frac{ds}{dt}
\]

The instantaneous rate of change of velocity with respect to time is called the **acceleration** \( a(t) \) of the object. The acceleration function is the derivative of the velocity function, the second derivative of the position function:

\[
a(t) = v'(t) = s''(t) \quad \text{or} \quad a = \frac{dv}{dt} = \frac{d^2s}{dt^2}
\]

**Example:** A motorcycle starts from rest and the graph of its position function is shown below, where \( s \) is measured in feet and \( t \) in seconds. Use it to graph the velocity and acceleration of the car.

What is the acceleration at \( t = 3 \) seconds?
The third derivative \( f''' \) is the derivative of the second derivative: \( f''' = (f'')' \) and can be interpreted as the slope or rate of change of the curve \( y = f''(x) \).

\[
y''' = f'''(x) = \frac{d}{dx} \left( \frac{d^2 y}{dx^2} \right) = \frac{d^3 y}{dx^3}
\]

Continuing the process, we have \( y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n} \).

**Example:** If \( f(x) = x^2 - x \), find \( f'''(x) \) and \( f^{(4)}(x) \).

The third derivative of the position function is the derivative of the acceleration function and is called the **jerk**. It is the rate of change of acceleration; a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.