Math 150 Lecture Notes
Rational Functions

A **rational function** is a function of the form $r(x) = \dfrac{P(x)}{R(x)}$ where P and Q are polynomials.

The **domain of a rational function** consists of all real number x except those for which the denominator is zero.

The line $x = a$ is a **vertical asymptote** of the function f if $f(x)$ approaches $\pm\infty$ as x approaches a from the right or left.

The line $y = b$ is a **horizontal asymptote** of the function f if $f(x)$ approaches b as x approaches $\pm\infty$.

Asymptotes of Rational Functions

Let r be the rational function $r(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots b_1 x + b_0}$

1. The vertical asymptotes of r are the lines $x = a$ where a is a zero of the denominator.

2. a. If $n < m$, then r has horizontal asymptote $y = 0$.

   b. If $n = m$, then r has horizontal asymptote $y = \dfrac{a_n}{b_m}$.

   c. If $n > m$, then r has no horizontal asymptote.

To Sketch a Graph of a Rational Function

1. Factor the numerator and denominator. Note that a factor common to numerator and denominator indicates where there is a "hole" in the graph.
2. Find x- and y-intercepts and graph them.
3. Find any vertical asymptotes and graph them with dotted lines.
4. Find the horizontal/oblique asymptote and graph it with a dotted line.
5. Make a table of values including test points to determine behavior near asymptotes and plot additional points as needed to determine the rest of the graph.

Example 1: Sketch the graph: $r(x) = \dfrac{x^3 - x^2}{x^3 - 3x - 2}$

Example 2: Sketch the graph: $r(x) = \dfrac{x^3 + 4}{2x^2 + x - 1}$

Example 3: Sketch the graph: $r(x) = \dfrac{-x^4 + 2x^3 - 2x}{(x-1)^2}$