Elementary methods for identifying linear asymptotes

Asymptotes of many elementary functions can be found without the explicit use of limits (although the derivations of such methods typically use limits).

Rational functions

A rational function has at most one horizontal asymptote or oblique (slant) asymptote, and possibly many vertical asymptotes.

The degree of the numerator and degree of the denominator determine whether or not there are any horizontal or oblique asymptotes. The cases are tabulated below, where deg(numerator) is the degree of the numerator, and deg(denominator) is the degree of the denominator.

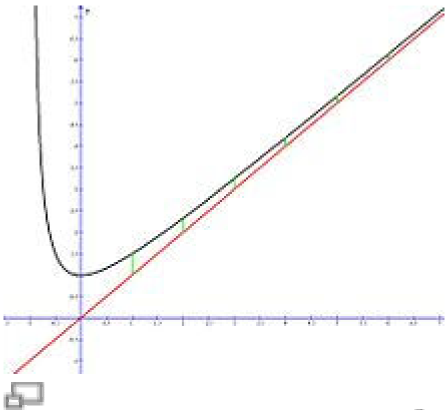
Table listing the cases of horizontal and oblique asymptotes for rational functions

deg(numerator) - deg(denominator)	**Horizontal/oblique asymptotes**	**Example, asymptote**
<0	*y*=0	$\frac{1}{x^2+1}, y=0$
0	y="ratio of leading coefficients"	$\frac{2x^2+7}{3x^2+x+12}, y=\frac{2}{3}$
1	1 oblique	$\frac{2x^3}{3x^2+1}, y=\frac{2}{3}x$
>1	None	$\frac{2x^4}{3x^2+1}$, none

The vertical asymptotes occur only when the denominator is zero (If both the numerator and denominator are zero, the multiplicities of the zero are compared). For example, the following function has vertical asymptotes at x=0, and x=1, but not at x=2

$$f(x)=\frac{x^2-5x+6}{x^3-3x^2+2x}=\frac{(x-2)(x-3)}{x(x-1)(x-2)}$$

Oblique asymptotes



Black: the graph of $f(x) = \frac{x^2+x+1}{x+1}$. Red: the asymptote $y = x$. Green: difference between the graph and its asymptote for $x = 1,2,3,4,5,6$

When the numerator of a rational function has degree exactly one greater than the denominator, the function has an oblique (slant) asymptote. The asymptote is the polynomial term after dividing the numerator and denominator. This phenomenon occurs because when dividing the fraction, there will be a linear term, and an error term. For example, consider the function

$$f(x) = \frac{x^2+x+1}{x+1} = x + \frac{1}{x+1}$$

shown to the right. As the value of x increases, f approaches the asymptote $y=x$. This is because the other term, $y=1/(x+1)$ becomes smaller.

If the degree of the numerator is more than 1 larger than the degree of the denominator, there will generally still be an error term that goes to zero as x increases, but the quotient will not be linear, and the function does not have an oblique asymptote.

The error term need not be so simple, however, as in this example.

$$\frac{2x^3}{3x^2+1} = \frac{2}{3}x - \frac{2x}{9x^2+3} \approx \frac{2}{3}x, \text{ for large } |x|.$$

http://en.wikipedia.org/wiki/Asymptote