ENUMERATIVE GEOMETRY OF PLANE CURVES

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Exercise 1. (a) Prove that a conic in $\mathbb{C}P^2$ having a singular point is the union of two lines.
(b) Prove that a cubic in $\mathbb{C}P^2$ having two singular points contains a line.
(c) Prove that a cubic in $\mathbb{C}P^2$ having three singular points is the union of three lines.

Exercise 2. We denote by $[x]$ the integer part of the number x. Deduce from Kontsevich formula that $2^{\left[\frac{d-1}{2}\right]}$ divides $N(d, 0)$.

Exercise 3 (Computation by hand of $N(3, 0)$ and $W(3)$)**.** We denote by Π_3 the space of cubic curves in $\mathbb{C}P^2$, recall that this is a projective space of dimension 9. Let $\mathcal{P} = \{x_1, \ldots, x_8\}$ be a set of 8 points in general position in $\mathbb{C}P^2$, and let $\mathcal{L}_\mathcal{P} \subset \Pi_3$ be the set of all cubics containg the set $\mathcal{P}$.

(a) Show that $\mathcal{L}_\mathcal{P}$ is a line in Π_3. Deduce that there exists a point $x_9 \in \mathbb{C}P^2 \setminus \mathcal{P}$ such that x_9 is contained in all cubics in $\mathcal{L}_\mathcal{P}$, and any two cubics in $\mathcal{L}_\mathcal{P}$ intersect exactly in $x_1, \ldots, x_9$.
(b) Let Σ be the blow-up of $\mathbb{C}P^2$ at the points $x_1, \ldots, x_9$. Prove that the Euler characteristic of Σ is 12.
(c) Show that there exists a natural map $\Sigma \to \mathcal{L}_p$. Deduce that the Euler characteristic of Σ is $N(3, 0)$.
(d) Adapt this computation in the real case to find $W(3) = 8$.
(e) Prove that there exists a configuration $\mathcal{P}$ such that all the 12 rational cubics that contain $\mathcal{P}$ are real.

Exercise 4 (Compare with Exercise 1)**.** A *crossing point* of a tropical curve C in $\mathbb{R}^2$ is a vertex of C whose dual polygon is a parallelogram.

(a) Prove that a tropical conic in $\mathbb{R}^2$ having crossing point is the union of two tropical lines.
(b) Prove that a tropical cubic in $\mathbb{R}^2$ having two crossing points contains a tropical line.
(c) Prove that a tropical cubic in $\mathbb{R}^2$ having three crossing points is the union of three tropical lines.

Exercise 5. Show that either 9 or 10 distinct rational tropical cubics pass through a given generic configuration of 8 points in $\mathbb{R}^2$.

Exercise 6. Prove that

$$N(d, 0) = W(d) \mod 4 \quad \forall d \geq 1.$$

Exercise 7. Using Figures 1, 2 and 3, compute

$$N(4, 3) = 1, \qquad N(4, 2) = 27, \qquad N(4, 1) = 225, \qquad N(4, 0) = 620, \quad \text{and} \quad W(4) = 240.$$

Date: May 29, 2017.

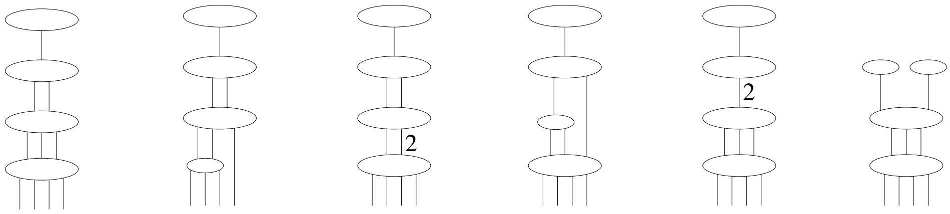

FIGURE 1. Floor diagrams of degree 4 and genus 3 or 2

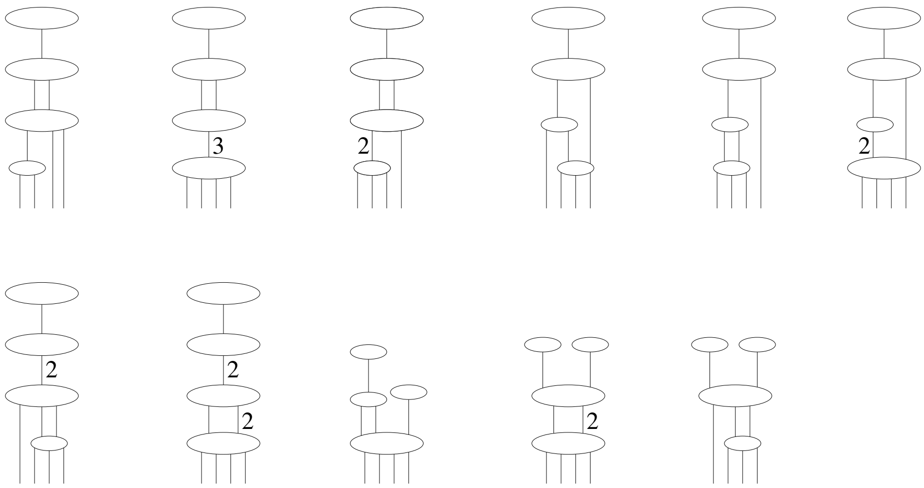

FIGURE 2. Floor diagrams of degree 4 and genus 1

Exercise 8. Using floor diagrams, prove that

$$N(d, \frac{(d-1)(d-2)}{2} - 1) = 3(d-1)^2,$$

and that

$$N(d, \frac{(d-1)(d-2)}{2} - 2) = \frac{3}{2}(d-1)(d-2)(3d^2 - 3d - 11).$$

Exercise 9 (Compare with Exercise 7)**.** Compute

$$G_{4,2} = 3q^{-1} + 21 + 3q,$$
$$G_{4,1} = 3q^{-2} + 33q^{-1} + 153 + 33q + 3q^2,$$
$$G_{4,0} = q^{-3} + 13q^{-2} + 94q^{-1} + 404 + 94q + 13q^2 + q^3.$$

Exercise 10 (Compare with Exercise 8)**.** Prove that

$$G_{d,\frac{(d-1)(d-2)}{2}} = 1 \quad \text{and} \quad G_{d,\frac{(d-1)(d-2)}{2}-1} = (d-1) \cdot \left(\frac{d-2}{2} \cdot q^{-1} + 2d - 1 + \frac{d-2}{2} \cdot q\right).$$

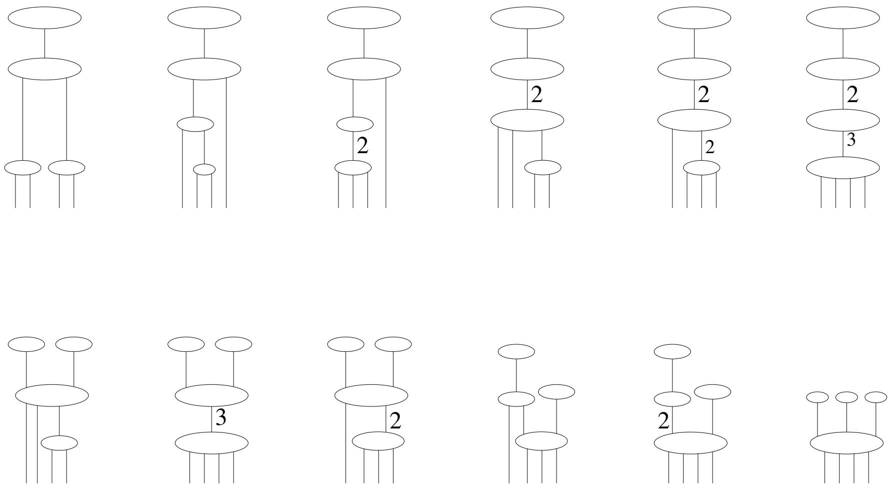

FIGURE 3. Floor diagrams of degree 4 and genus 0

Exercise 11. Fix a positive integer d and an integer $0 \leq g \leq \frac{(d-1)(d-2)}{2}$. Prove that the highest power of q which appears in $G_{d,g}$ is $\frac{(d-1)(d-2)}{2} - g$, and that the coefficient of the corresponding monomial of $G_{d,g}$ is equal to

$$\binom{\frac{(d-1)(d-2)}{2}}{g}.$$

Exercise 12. Fix a non-negative integer g and a positive integer k. Show that for any sufficiently large integer d and any generic collection $\mathcal{P}$ of $3d-1+g$ points in $\mathbb{R}^2$, there exists an irreducible tropical curve C of degree d and genus g in $\mathbb{R}^2$ such that C passes through the points of $\mathcal{P}$, and has a complex multiplicity greater than k.

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