Tensors: Eigenvectors

Third Session with Bernd Sturmfels at Ibadan

A d-dimensional tensor F of format $n \times n \times \cdots \times n$ can be written as a list of n tensors of dimension $d-1$ and format $n \times \cdots \times n$, each obtained by fixing the first index. The symmetrization of each of these n tensors is a homogeneous polynomial f_i of degree $d-1$ in n variables. The list of polynomials $f = (f_1, \ldots, f_n)$ defines a polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$. A vector $v \in \mathbb{C}^n$ is an *eigenvector* of the tensor F if $f(v) = \lambda \cdot v$ for some $\lambda \in \mathbb{C}$. We refer to a point $v \in \mathbb{P}^{n-1}$ as an *eigenvector* of F if it is a fixed point or a base point of the induced rational map $f : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$.

Question 1: Compute the eigenvectors of the Ibadan tensor. Repeat after permuting coordinates.

Question 2: Each symmetric tensor is identified with a homogeneous polynomial g of degree d in n variables. Its gradient ∇g is the vector of partial derivatives. Show that the eigenvectors of g are the fixed points or base points the *gradient map* $\nabla g : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$. What are the base points ?

Question 3: Write the cubic $g = x^3 + y^3 + z^3$ as a tensor. Compute its eigenvectors in $\mathbb{P}^2$.

Question 4: Write the quartic $g = xyz(x+y+z)$ as a tensor. Compute its eigenvectors in $\mathbb{P}^2$.

Question 5: Are all configurations of 13 points in $\mathbb{P}^2$ the eigenvectors of some (symmetric) tensor?

Question 6: How to maximize or minimize a function on the unit sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$? Compute all local minima and local maxima on $\mathbb{S}^2$ of the functions in # 3, # 4.

Question 7: Compute all eigenvectors in $\mathbb{P}^3$ of the quartic $g = x^4 + y^4 + z^4 + w^4$.

Question 8: Compute all eigenvectors in $\mathbb{P}^3$ of the quintic $g = xyzw(x+y+z+w)$.

Question 9: What is the maximal number of isolated **real** eigenvectors for a symmetric tensor? Answer this question for all values of d and n that were encountered in the examples above.