Tensors: Equations for Low Rank
Fourth Session with Bernd Sturmfels at Ibadan

The closure of the set of all tensors of rank $\leq r$ is a variety in the projective space $\mathbb{P}(\mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \cdots \times \mathbb{C}^{n_d})$. The closure of the set of all symmetric tensors of rank $\leq r$ is a variety in the projective space $\mathbb{P}(\mathrm{Sym}_d(\mathbb{C}^n))$. Here we mean the complex rank of a complex tensor.

Question 1: What are the dimensions of these projective spaces?

Question 2: What are these varieties called in text books on algebraic geometry?

Question 3: Find formulas for the dimension and degree of these varieties when $r = 1$.

Question 4: Study the variety of $2 \times 2 \times 3$-tensors of rank ≤ 2. Compute its prime ideal.

Question 5: Study the variety of $2 \times 3 \times 3$-tensors of rank ≤ 2. Compute its prime ideal.

Question 6: Determine the degree and the singular locus of the two varieties in #4 and # 5.

Question 7: What is the rank of random tensor of format $4 \times 4 \times 4 \times 4$?

Question 8: Study the variety of $3 \times 3 \times 3$-tensors of rank ≤ 3. Compute its prime ideal.

Question 9: Consider symmetric tensors of rank ≤ 3 for $d = n = 3$. Compute their prime ideal.

Question 10: Does the hyperdeterminant of a given format vanish on all tensors of low rank?

Question 11: For a matrix, the rank is the number of nonzero eigenvalues. How about tensors?

Question 12: What is the rank of random symmetric tensor? Find a formula in terms of d and n.