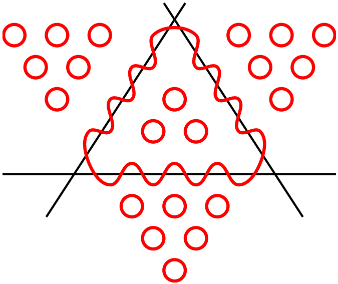
Non-Existence of Torically Maximal Hypersurfaces

Kristin Shaw

Technische Universität Berlin & The Fields Institute

CMS Winter Meeting

December 2016



Maximal curves in $\mathbb{R}P^2$

Let $F \in \mathbb{R}[x_0, x_1, x_2]$ be homogeneous s.t. $C := V(F) \subset \mathbb{C}P^2$ is smooth.

Then $\mathbb{R}C := C \cap \mathbb{R}P^2$ is a disjoint union of embedded circles.

Some real curves of degree 4 in $\mathbb{R}P^2$

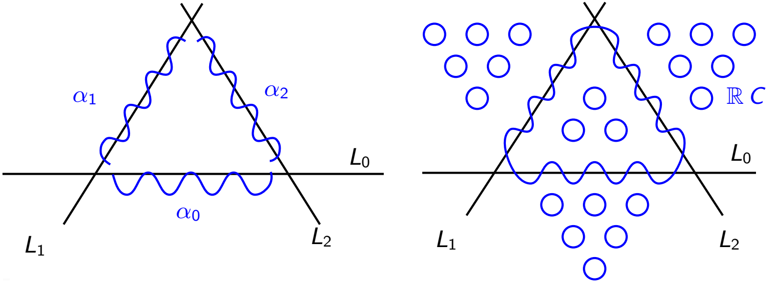
Theorem (Harnack 1876)

If F is homogeneous of degree d then

$$\dim H_0(\mathbb{R}C) \leq \frac{(d-1)(d-2)}{2} + 1.$$

Any curve obtaining this upper bound is called **maximal** or an M-**curve**.

Harnack's M-curve construction



A **simple Harnack curve** has 3 disjoint arcs $\alpha_0, \alpha_1, \alpha_2 \subset \mathbb{R}C$ such that

$$|\alpha_i \cap L_i| = d \quad \text{for} \quad i = 0, 1, 2, \quad \text{and lines} \quad L_0, L_1, L_2 \subset \mathbb{C}P^2.$$

Theorem (Mikhalkin 2000)

If $C \subset \mathbb{C}P^2$ is a simple Harnack curve then the topology of the triad $(\mathbb{R}P^2; \mathbb{R}C, \cup L_i)$ is unique.

Properties of simple Harnack curves

Theorem

A smooth curve $C \subset \mathbb{C}P^2$ a simple Harnack curve ifff:

- *maximal in every affine chart (Mikhalkin 2000);*
- *The amoeba of $\mathbb{R}C$ has no inflection points (Mikhalkin 2000);*
- *The amoeba map is at most $2:1$ (Mikhalkin-Rullgård 2001);*
- *The amoeba of C has maximal area (Mikhalkin-Rullgård 2001);*
- *The amoeba of $\mathbb{R}C$ has maximal curvature (Passare-Risler 2010);*
- *The* **log Gauß map** $\gamma : C \to \mathbb{C}P^1$ *is* **totally real** *(Passare-Risler 2010).*

Torically maximal hypersurfaces in $(\mathbb{C}^*)^n$ were defined by Mikhalkin (2001).

No known examples when $n > 2$ except for hyperplanes!

The logarithmic Gauß map

Let $V \subset (\mathbb{C}^*)^n$ be defined by $F \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with Newton polytope Δ.

Definition (Kapranov 1991)

The **log Gauß map** $\gamma : V \to \mathbb{C}P^{n-1}$ is

$$\gamma(z_1, \dots, z_n) \mapsto [z_1 \frac{\partial F}{\partial z_1} : \cdots : z_n \frac{\partial F}{\partial z_n}].$$

- The log Gauß map is the coordinatewise logarithm map composed with the usual Gauß map;
- A-discriminantal varieties characterised by log Gauß map (Kapranov 1991);
- The graph of the log Gauß map is the **maximum likelihood variety** $\mathcal{X} \subset (\mathbb{C}^*)^n \times \mathbb{C}P^{n-1}$.

The log Gauß map extends to $\overline{V} \subset \mathrm{TV}(\Delta)$ where $\mathrm{TV}(\Delta)$ is toric variety of Δ.

Torically maximal hypersurfaces in $(\mathbb{C}^*)^n$

Definition

A map between real varieties $f : X \to Y$ is:

- **totally real** if $f^{-1}(y) \subset \mathbb{R}X$ for all $y \in \mathrm{Im}(f) \cap \mathbb{R}Y$;
- **generically totally real** if $f^{-1}(y) \subset \mathbb{R}X$ for all $y \in \mathrm{Im}(f) \cap \mathbb{R}Y$ outside of a codimension 1 subset.

Definition

Let $V \subset (\mathbb{C}^*)^n$ be a $\mathbb{R}$-hypersurface and $\overline{V}$ be its closure in $\mathrm{TV}(\Delta)$,

- V is **torically maximal** if $\gamma : \overline{V} \to \mathbb{C}P^{n-1}$ is generically totally real;
- V is **strongly torically maximal** if $\gamma : \overline{V} \to \mathbb{C}P^{n-1}$ is totally real.

Non-existence of torically maximal hypersurfaces

Assume that $V \subset (\mathbb{C}^*)^n$, $\overline{V} \subset \mathrm{TV}(\Delta)$, and $\overline{V} \cap \mathrm{TV}(\Delta_i)$ are all non-singular.

Theorem 1 (BMRS)

If $n \geq 3$ and $V \subset (\mathbb{C}^*)^n$ is a **torically maximal** *hypersurface such that $TV(\Delta) = \mathbb{C}P^n$ then $\overline{V}$ is a hyperplane.*

Theorem 2 (BMRS)

If $n \geq 3$ and $V \subset (\mathbb{C}^*)^n$ is a **strongly torically maximal hypersurface** *then $\overline{V} \subset TV(\Delta)$ is a hyperplane in projective space.*

Proof of Theorem 1

Theorem (BMRS)

If $V \subset (\mathbb{C}^*)^n$ is a hypersurface with $TV(\Delta) = \mathbb{C}P^n$ then the log Gauß map $\gamma : \overline{V} \to \mathbb{C}P^n$ has **finite fibres***.*

Proof of Theorem 1 (n = 3).

If V is torically maximal $\Rightarrow$ V strongly torically maximal $\Rightarrow$

$\gamma : \mathbb{R}\overline{V} \to \mathbb{R}P^2$ is a covering map (Kummer-Shamovich 2015) $\Rightarrow$

$$\mathbb{R}\overline{V} = k(S^2) \sqcup l(\mathbb{R}P^2).$$

$\forall$ coordinate hyperplane $H_i \subset \mathbb{C}P^3$, $\overline{V} \cap H_i$ is a deg d simple Harnack curve and

$$\deg(\gamma|_{\mathcal{O}_i}) = 3d - 2.$$

But $\deg(\gamma|_{\mathcal{O}_i}) = 1$ or 2 since $\mathcal{O}_i \subset S^2$ or $\mathbb{R}P^2 \Rightarrow d = 1$ and $\overline{V}$ must be a hyperplane. □

Singular and half dimensional examples

Example

If $F(z) = az_3^2 + z_3 + z_2 + z_1 + 1$ and $a \in (0, \frac{1}{4})$ then $\gamma : \overline{V} \to \mathbb{C}P^2$ is totally real.

What are the singular (strongly) torically maximal hypersurfaces?

Example

If $V = C_1 \times C_2 \subset (\mathbb{C}^*)^4$ where C_1, C_2 Harnack curves, then $\gamma : \overline{V} \to \mathrm{Gr}(2,4)$ is totally real and $\gamma(\overline{V}) = \mathbb{C}P^1 \times \mathbb{C}P^1$ has real structure $\mathbb{R}P^1 \times \mathbb{R}P^1$.

What are the (strongly) torically maximal varieties of arbitrary codimension?

Thank you!