

TWO MURNAGHAN-NAKAYAMA RULES IN SCHUBERT CALCULUS

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ABSTRACT. The Murnaghan-Nakayama rule expresses the product of a Schur function with a Newton power sum in the basis of Schur functions. We establish a version of the Murnaghan-Nakayama rule for Schubert polynomials and a version for the quantum cohomology ring of the Grassmannian. These rules compute all intersections of Schubert cycles with tautological classes coming from the Chern character. Like the classical rule, both rules are multiplicity-free signed sums.

1. INTRODUCTION

Each integer partition λ has an associated Schur symmetric function s_λ , and these Schur functions form a basis for the \mathbb{Q} -algebra of symmetric functions, which is also freely generated by the power sum symmetric functions p_r . The Murnaghan-Nakayama rule is the expansion in the Schur basis of the product by a power sum,

$$p_r \cdot s_\lambda = \sum_{\mu} (-1)^{\text{ht}(\mu/\lambda)+1} s_{\mu},$$

the sum over all partitions μ such that μ/λ is a rim hook of size r and $\text{ht}(\mu/\lambda)$ is the height (number of rows) of μ/λ .

Products $p_\lambda := p_{\lambda_1} \cdots p_{\lambda_k}$ of power sums form another basis for symmetric functions, and the change of basis matrix between these two is the character table for the symmetric group. In this way, the Murnaghan-Nakayama rule gives a formula for the characters of the symmetric group [21, 22, 23].

The cohomology ring of the Grassmannian $\text{Gr}(k, n)$ of k -planes in n -space has a basis of Schubert cycles σ_λ for λ a partition with $\lambda_1 \leq n-k$ and $\lambda_{k+1} = 0$ (written $\lambda \leq \square_{k,n}$). Its multiplication is induced from the ring of symmetric functions under the map that sends a Schur function s_λ to σ_λ when $\lambda \leq \square_{k,n}$ and otherwise sends it to 0. Images of the power sum functions give the Chern characters of the tautological bundle. Thus the Murnaghan-Nakayama rule computes intersections of Schubert cycles with tautological classes.

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We extend this classical formula in two directions, to Schubert polynomials [16] and to the quantum cohomology of Grassmannians [6]. For each permutation $w \in S_n$, let \mathfrak{S}_w be the Schubert polynomial of Lascoux and Schützenberger [15].

Theorem 1. *Let $k, r < n$ be positive integers and $w \in S_n$ a permutation. Then*

$$p_r(x_1, \dots, x_k) \cdot \mathfrak{S}_w = \sum (-1)^{\text{ht}(\eta)+1} \mathfrak{S}_{w\eta},$$

the sum over all $(r+1)$ -cycles η such that

$$w <_k w\eta \quad \text{with} \quad \ell(w\eta) = \ell(w) + r,$$

where $<_k$ is the k -Bruhat order and $\text{ht}(\eta) = \#\{i \leq k \mid \eta(i) \neq i\}$.

For simplicity, our notation intentionally suppresses the dependence of $\text{ht}(\eta)$ on k .

The cohomology of the manifold $\mathbb{F}\ell(n)$ of complete flags in \mathbb{C}^n has a basis of Schubert cycles $[X_w]$ indexed by permutations w of $1, \dots, n$. Mapping the variables x_1, \dots, x_n to the Chern roots of the dual of the tautological flag bundle is a surjection onto the cohomology ring with the Schubert polynomial \mathfrak{S}_w sent to the Schubert cycle $[X_w]$ and power sums sent to tautological classes. Thus this Murnaghan-Nakayama rule also computes intersections of Schubert cycles with tautological classes.

Additively, the quantum cohomology $qH^*(\text{Gr}(k, n))$ of the Grassmannian $\text{Gr}(k, n)$ is the vector space of polynomials in a parameter q with coefficients from $H^*(\text{Gr}(k, n))$. It has a new multiplication $*$ among Schubert cycles that encodes the three-point Gromov-Witten invariants and reduces to the usual one when $q = 0$ [6].

Theorem 2. *Let $k, r < n$ be positive integers and $\lambda \leq \square_{k,n}$. Then*

$$p_r * \sigma_\lambda = \sum_{\mu} (-1)^{\text{ht}(\mu/\lambda)+1} \sigma_\mu - (-1)^k q \sum_{\nu} (-1)^{\text{ht}(\lambda/\nu)+1} \sigma_\nu,$$

where the first sum is over all $\mu \leq \square_{k,n}$ with μ/λ a rim hook of size r and the second sum is over all $\nu \leq \lambda$ with λ/ν a rim hook of size $n-r$.

The Murnaghan-Nakayama rule has many generalizations. Fomin and Green gave a version for non-commutative symmetric functions, which led to formulas for characters of representations associated to stable Schubert and Grothendieck polynomials [9]. McNamara gave a skew version [18], which Konvalinka generalized to a skew rule for multiplication by a ‘quantum’ (perturbed by a parameter q) power sum function [14]. Bandlow, et al. gave a version in the cohomology of an affine Grassmannian [1]. Tewari gave a version for noncommutative Schur functions [29]. Wildon gave a plethystic version [32]. Ross gave a version for loop Schur functions [25], providing a fundamental step in the orbifold Gromov-Witten/Donaldson-Thomas correspondence [26].

This paper is organized as follows. In Section 2 we recall some aspects of the Schur basis of symmetric functions and the classical Murnaghan-Nakayama rule. In Section 3, we recall some results concerning the Schubert polynomials, prove Theorem 1, and explain its significance for the flag manifold. We conclude with Section 4, where we recall the quantum cohomology ring of the Grassmannian and prove Theorem 2.

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2. THE CLASSICAL MURNAGHAN-NAKAYAMA RULE

For a complete and elegant treatment of symmetric functions see Macdonald's book [17]. The \mathbb{Q} -algebra Λ of homogeneous symmetric functions in countably many indeterminants x_1, x_2, \dots is freely generated by three distinguished sequences of symmetric functions. One consists of the elementary symmetric functions, e_a , where e_a is the formal sum of all square-free monomials of degree a . A second consists of the complete symmetric functions, h_b , where h_b is the formal sum of all monomials of degree b . The third distinguished sequence consists of the power sum functions, p_r , where p_r is the formal sum of the r th powers of the variables, $p_r = x_1^r + x_2^r + \dots$.

The most important family of symmetric functions are the Schur functions s_λ , which are indexed by partitions $\lambda: \lambda_1 \geq \dots \geq \lambda_k \geq 0$, where λ_i is an integer. The Schur function s_λ may be defined by the Jacobi-Trudi formula as a determinant

$$s_\lambda = \det (h_{\lambda_i + j - i})_{i,j=1}^k.$$

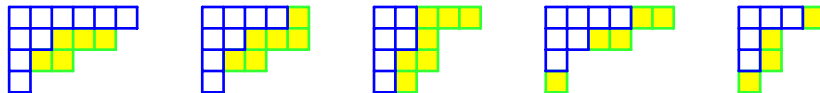
This has degree $|\lambda| := \lambda_1 + \dots + \lambda_k$. We have that $s_{(b)} = h_b$ and $s_{(1^a)} = e_a$, where (1^a) is a sequence of a 1s. The set of all Schur functions forms a basis for Λ .

We often represent a partition λ by its Young diagram, which is a left-justified array of boxes with λ_i boxes in row i . For example

$$(3, 1, 0, 0) \longleftrightarrow \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \quad \text{and} \quad (5, 4, 3, 1) \longleftrightarrow \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \\ \hline \square & \square & \square & & \\ \hline \square & & & & \\ \hline \end{array}.$$

When λ and μ are partitions with $\lambda_i \leq \mu_i$ for all i , we write $\lambda \leq \mu$ and write μ/λ for the set-theoretic difference $\mu \setminus \lambda$ of their diagrams. The *size* of μ/λ is its number of boxes.

A *rim hook* is a skew shape μ/λ that meets a connected set of Northwest to Southeast (\searrow) diagonals with no two boxes in the same diagonal. The *height* $ht(\mu/\lambda)$ of a rim hook μ/λ is its number of rows. A *horizontal strip* μ/λ has no two boxes in the same column, and a *vertical strip* μ/λ has no two boxes in the same row. Below are rim hooks of height two, three, and four of sizes five, six, and seven, respectively, as well as a horizontal strip of size five and a vertical strip of size four.



A partition λ is a *hook* if $\lambda_2 \leq 1$. For $a, b \geq 1$, the hook with a rows and b columns is $(b, 1^{a-1})$. Since $e_{a-1} \cdot h_b = s_{(b, 1^{a-1})} + s_{(b+1, 1^{a-2})}$, we have

$$(2.1) \quad s_{(b, 1^{a-1})} = e_{a-1} \cdot h_b - e_{a-2} \cdot h_{b+1} + \dots + (-1)^{a-1} h_{a+b-1}.$$

The power sum symmetric function p_r is the alternating sum of hooks of size r ,

$$(2.2) \quad p_r = h_r - s_{(r-1, 1)} + \dots + (-1)^{r-1} e_r = \sum_{i=0}^{r-1} (-1)^i s_{(r-i, 1^i)}.$$

We give formulas for the product of a Schur function by these generating sets.

Proposition 3. *Let λ be a partition and a, b, r be positive integers. Then we have*

- (i) $e_a \cdot s_\lambda = \sum s_\mu$, the sum over all μ with μ/λ a vertical strip of size a
- (ii) $h_b \cdot s_\lambda = \sum s_\mu$, the sum over all μ with μ/λ a horizontal strip of size b .
- (iii) $p_r \cdot s_\lambda = \sum (-1)^{\text{ht}(\mu/\lambda)+1} s_\mu$, the sum over all μ with μ/λ a rim hook of size r .

Formulas (i) and (ii) of Proposition 3 are called Pieri rules, for they are essentially due to Pieri [24], and Formula (iii) is the Murnaghan-Nakayama rule.

These formulas make sense in two important quotients of Λ . The ring Λ_k of polynomials that are symmetric in x_1, \dots, x_k for an integer $k > 0$ is obtained from Λ by specializing $x_a = 0$ (equivalently $e_a = 0$) for $a > k$. This also has the vector space presentation

$$\Lambda_k \simeq \Lambda/\mathbb{Q}\{s_\lambda \mid \lambda_{k+1} > 0\}.$$

Thus Λ_k has a basis of Schur polynomials $s_\lambda(x_1, \dots, x_k)$ where λ has at most k parts in that $\lambda_{k+1} = 0$. It follows that the Murnaghan-Nakayama rule holds in Λ_k , if we restrict to partitions with at most k parts.

Let $n > k$ be an integer. The *Grassmannian* $\text{Gr}(k, n)$ of k -planes in \mathbb{C}^n is a complex manifold of dimension $k(n-k)$. Its cohomology ring with \mathbb{Q} -coefficients is a quotient of both Λ_k and Λ ,

$$H^*(\text{Gr}(k, n)) \simeq \Lambda/\langle e_a, h_b \mid a > k, b > n-k \rangle = \Lambda/\mathbb{Q}\{s_\lambda \mid \lambda \not\leq \square_{k,n}\},$$

where $\square_{k,n}$ is the partition with k parts, each of size $n-k$. We also have that

$$(2.3) \quad H^*(\text{Gr}(k, n)) \simeq \Lambda_k/\langle h_{n-k+1}, \dots, h_{n-1}, h_n \rangle.$$

By these presentations, the images σ_λ of the Schur functions s_λ for $\lambda \leq \square_{k,n}$ form a basis for $H^*(\text{Gr}(k, n))$. These σ_λ are called *Schubert cycles* as they are Poincaré dual to the fundamental cycles in homology of Schubert varieties. The Pieri and Murnaghan-Nakayama rules hold in $H^*(\text{Gr}(k, n))$, if we restrict to partitions $\lambda, \mu \leq \square_{k,n}$, and to $a \leq k$, $b \leq n-k$, and $r < n$.

The homomorphism $\psi: \Lambda_k \rightarrow H^*(\text{Gr}(k, n))$ implicit in the quotient (2.3) may be understood geometrically as follows. The tautological bundle $E \rightarrow \text{Gr}(k, n)$ is the subbundle of $\mathbb{C}^n \times \text{Gr}(k, n)$ whose fibre over a point $H \in \text{Gr}(k, n)$ is the k -plane H . The map ψ sends the elementary symmetric polynomial e_a to the a th Chern class $c_a(E^\vee)$ of the dual of E , equivalently, it sends the variables x_1, \dots, x_k to the Chern roots y_1, \dots, y_k of E^\vee .

The Chern character $\text{ch}(E^\vee) \in H^*(\text{Gr}(k, n))$ is the cohomology class

$$\text{ch}(E^\vee) = \sum_{i=1}^k \exp(y_i) = \sum_{i=1}^k 1 + y_i + \frac{y_i^2}{2} + \frac{y_i^3}{3!} + \dots$$

In terms of the map ψ , it is

$$\psi(k + p_1 + \frac{1}{2}p_2 + \frac{1}{3!}p_3 + \dots) = k + \sum_{i=1}^{n-1} \frac{1}{i!} p_i(y_1, \dots, y_k).$$

The classes $\text{ch}_r(E^\vee) = \frac{1}{r!} p_r(y_1, \dots, y_k)$ are the *tautological classes*. Thus the Murnaghan-Nakayama rule is a formula in cohomology for multiplication by tautological classes.

3. MURNAGHAN-NAKAYAMA RULE FOR SCHUBERT POLYNOMIALS

We prove Theorem 1 using results about multiplication of Schubert polynomials. An excellent reference for Schubert polynomials, with proofs, is Macdonald's book [16].

The symmetric group S_n acts on polynomials in x_1, \dots, x_n by permuting the variables. Write (i, j) for the transposition interchanging i and j with $i < j$ and t_i for the simple transposition $(i, i+1)$. Permutations $w \in S_n$ are products of simple transpositions, $w = t_{a_1} \cdots t_{a_m}$. The minimal length of such a factorization is when $m = \ell(w)$, the number of inversions in w . In this case, (a_1, \dots, a_m) is a *reduced word* for w .

The divided difference operator for $i = 1, \dots, n-1$ is

$$\partial_i := \frac{1 - t_i}{x_i - x_{i+1}},$$

which acts on the polynomial ring $\mathbb{Q}[x_1, \dots, x_n]$. These satisfy $\partial_i \circ \partial_i = 0$ and the braid relations, so that if (a_1, \dots, a_m) is a reduced word for w , then $\partial_w := \partial_{a_1} \circ \cdots \circ \partial_{a_m}$ depends on w and not on the choice of reduced word. Building on work of Bernstein, Gelfand, and Gelfand [5] and of Demazure [8], Lascoux and Schützenberger [15] defined the Schubert polynomial \mathfrak{S}_w to be

$$\mathfrak{S}_w := \partial_{w^{-1}\omega_0}(x_1^{n-1}x_2^{n-2} \cdots x_{n-1}),$$

where ω_0 is the longest element of the symmetric group S_n , $\omega_0(i) = n+1-i$. Schubert polynomials are linearly independent. For example, $\mathfrak{S}_{t_k} = x_1 + \cdots + x_k$.

We may embed S_n into S_{n+m} as permutations fixing $n+1, \dots, n+m$. Schubert polynomials are stable in the sense that for $w \in S_n$ we get the same the Schubert polynomial regarding $w \in S_n$ as we do regarding $w \in S_{m+n}$. If S_∞ is the set of permutations of $\{1, 2, \dots\}$ that fix all but finitely many integers, then \mathfrak{S}_w is well-defined for $w \in S_\infty$, and these form a basis for the polynomial ring $\mathbb{Q}[x_1, x_2, \dots]$.

All Schur symmetric polynomials are Schubert polynomials. If w is a permutation with a unique descent at k , so that $w(i) > w(i+1)$ implies that $i = k$, then \mathfrak{S}_w equals the Schur polynomial $s_\lambda(x_1, \dots, x_k)$, where $\lambda = (w(k)-k, \dots, w(2)-2, w(1)-1)$.

An open problem is to give a combinatorial formula for the expansion of a product of Schubert polynomials in the basis of Schubert polynomials. This is possible as the coefficients are nonnegative, and it would be an analog of the Littlewood-Richardson rule. The first step was due to Monk [19],

$$(3.1) \quad \mathfrak{S}_{t_k} \cdot \mathfrak{S}_w = (x_1 + \cdots + x_k) \cdot \mathfrak{S}_w = \sum \mathfrak{S}_u,$$

the sum over all permutations $u = w(i, j)$ where $i \leq k < j$ and $\ell(u) = \ell(w) + 1$. The range of summation in (3.1) gives the cover relation \prec_k in the k -Bruhat order on the symmetric group. Expanding $(x_1 + \cdots + x_k)^m$ in the basis of Schur polynomials and iterating the formula (3.1) shows that any Schubert polynomial \mathfrak{S}_u appearing in the product of \mathfrak{S}_w with a symmetric polynomial in x_1, \dots, x_k must have $w \prec_k u$. When $w \prec_k w(i, j)$ with $i \leq k < j$, we write $w \xrightarrow{w(i)} w(i, j)$, labeling the cover by the value of w at i .

Monk's rule was generalized to a Pieri rule for multiplying by elementary or complete homogeneous symmetric polynomials, $e_a(x_1, \dots, x_k)$ and $h_b(x_1, \dots, x_k)$ in [28], and that paper also gave the formula for hook Schur polynomials.

Proposition 4 ([28], Theorem 8). *Let $a \leq k$ and b be positive integers and set $r = a+b-1$. For a permutation $w \in S_\infty$, we have*

$$s_{(b,1^{a-1})}(x_1, \dots, x_k) \cdot \mathfrak{S}_w = \sum \mathfrak{S}_{\text{end}(\gamma)}$$

the sum over all chains $\gamma: w \xrightarrow{\alpha_1} w^1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_r} w^r = \text{end}(\gamma)$ in the k -Bruhat order satisfying

$$(3.2) \quad \alpha_1 > \dots > \alpha_a < \alpha_{a+1} < \dots < \alpha_r.$$

A chain γ whose labels satisfy (3.2) is *peakless*.

Chains in the k -Bruhat order were studied in [2, 3]. For a permutation ζ , let $\text{up}(\zeta) := \{a \mid a < \zeta(a)\}$. If k is any integer with $k \geq |\text{up}(\zeta)|$, then there is a permutation w with $w <_k \zeta w$. Moreover, the interval $[w, \zeta w]_k$ in the k -Bruhat order between w and ζw , considered as a poset whose covers are labeled, is independent of w and k . In particular, the *rank* of ζ is $\text{rk}(\zeta) := \ell(\zeta w) - \ell(w)$ for any w, k with $w <_k \zeta w$. Two permutations ζ, η are *disjoint* if $\zeta\eta = \eta\zeta$ and $\text{rk}(\zeta\eta) = \text{rk}(\zeta) + \text{rk}(\eta)$ —this is equivalent to the supports of ζ and η forming a noncrossing partition of their union (see [2, § 3.3]). By *support* of a permutation ζ , we mean the set $\{i \mid \zeta(i) \neq i\}$.

In [4, § 6.2] permutations ζ in which the interval $[w, \zeta w]_k$ admits a peakless chain were studied. An $(r+1)$ -cycle ζ is *minimal* if it has the minimal possible rank r . Lemma 6.7 of [4] states that if ζ is a minimal cycle and $w <_k \zeta w$, then there is a unique peakless chain in $[w, \zeta w]_k$. Moreover, $|\text{up}(\zeta)|$ is the length of the decreasing subsequence in the labels of that chain (3.2), which is also equal to the *height* of $\eta := w^{-1}\zeta w$, which is $\text{ht}(\eta) := \#\{i \leq k \mid \eta(i) \neq i\}$. For simplicity, we suppress the dependence of height on k , as η itself depends upon w, k , and ζ in this discussion. We record these facts.

Proposition 5. *Let ζ be a minimal cycle, w be a permutation, and k be an integer such that $w <_k \zeta w$. Then there is a unique peakless chain in $[w, \zeta w]_k$. If (3.2) is the sequence of labels in this chain and we set $\eta := w^{-1}\zeta w$ so that $\zeta w = w\eta$ then*

$$|\text{up}(\zeta)| = a = \text{ht}(\eta).$$

A permutation is minimal if it is the product of disjoint minimal cycles. These results imply that a permutation ζ is minimal if and only if whenever $w <_k \zeta w$, there is a peakless chain in the interval $[w, \zeta w]_k$. We restate Theorem 1.

Theorem 1. *Let k, r be positive integers and $w \in S_\infty$ a permutation. Then*

$$p_r(x_1, \dots, x_k) \cdot \mathfrak{S}_w = \sum (-1)^{\text{ht}(\eta)+1} \mathfrak{S}_{w\eta},$$

the sum over all $(r+1)$ -cycles η such that

$$w <_k w\eta \quad \text{with} \quad \ell(w\eta) = \ell(w) + r.$$

Example 6. Before presenting a proof of Theorem 1, we give an example of the Murnaghan-Nakayama rule for Schubert polynomials. For this $k = r = 4$, and we restrict to permutations $w \in S_8$, expressing such a w as the word of its values $w(1) \dots w(8)$.

$$\begin{aligned} p_4(x_1, \dots, x_4) \cdot \mathfrak{S}_{34165278} &= \mathfrak{S}_{35671248} + \mathfrak{S}_{36471258} + \mathfrak{S}_{45362178} + \mathfrak{S}_{46173258} \\ &\quad - \mathfrak{S}_{34672158} - \mathfrak{S}_{34681257} - \mathfrak{S}_{36184257}. \end{aligned}$$

If $w = 34165278$, then these indices are

$$\begin{aligned} 35671248 &= w(3, 4, 7, 2, 5), & 36471258 &= w(2, 4, 7, 5, 3), & 45362178 &= w(1, 2, 5, 6, 4), \\ 46173258 &= w(1, 2, 4, 7, 5), & 34672158 &= w(3, 4, 7, 5, 6), & 34681257 &= w(3, 4, 8, 7, 5), \end{aligned}$$

and $36184257 = w(2, 4, 8, 7, 5)$. The first four have height 3, while the last three have height 2. This agrees with the signs in Theorem 1.

Proof of Theorem 1. Since $x_i = \mathfrak{S}_{t_i} - \mathfrak{S}_{t_{i-1}}$, Monk's rule gives the transition formula,

$$x_i \cdot \mathfrak{S}_w = \sum_{\substack{i < b \\ \ell(w(i,b)) = \ell(w)+1}} \mathfrak{S}_{w(i,b)} - \sum_{\substack{a < i \\ \ell(w(a,i)) = \ell(w)+1}} \mathfrak{S}_{w(a,i)}.$$

Thus if a Schubert polynomial \mathfrak{S}_u appears in $x_i^r \cdot \mathfrak{S}_w$, then $u = w\eta$, where η is an $(r+1)$ -cycle. Summing over $i = 1, \dots, k$ we see that if \mathfrak{S}_u appears in $p_r(x_1, \dots, x_k) \cdot \mathfrak{S}_w$, then $u = w\eta$ with η an $(r+1)$ -cycle. As $p_r(x_1, \dots, x_k)$ is symmetric, we have $w <_k w\eta$. Writing $w\eta = \zeta w$, then ζ is an $(r+1)$ -cycle of rank r , and is therefore a minimal cycle.

By Proposition 5, there is a unique peakless chain in $[w, \zeta w]_k$ and its sequence of decreasing labels has length $a := \text{ht}(\eta)$. By Proposition 4, the only product $s_{(r-i, 1^i)} \cdot \mathfrak{S}_w$ containing $\mathfrak{S}_{w\eta}$ is $s_{(r-a+1, 1^{a-1})} \cdot \mathfrak{S}_w$, and it occurs with multiplicity 1. By (2.2), the coefficient of $\mathfrak{S}_{w\eta}$ in $p_r \cdot \mathfrak{S}_w$ is $(-1)^{a-1} = (-1)^{\text{ht}(\eta)+1}$, which completes the proof. \square

Remark 7. We apply the Murnaghan-Nakayama rule to the cohomology of flag manifolds. Fix a positive integer n . The flag manifold $\mathbb{F}\ell(n)$ is the collection of all complete flags,

$$F_\bullet = 0 \subset F_1 \subset F_2 \subset \dots \subset F_n = \mathbb{C}^n,$$

where the subspace F_i has dimension i . There is a universal family of flags over $\mathbb{F}\ell(n)$,

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}_n = \mathbb{C}^n \times \mathbb{F}\ell(n),$$

where \mathcal{F}_i is the rank i tautological subbundle given by the i th flag. For each $i = 1, \dots, n$, let $y_i := -c_1(\mathcal{F}_i/\mathcal{F}_{i-1})$. These classes generate the cohomology of $\mathbb{F}\ell(n)$ with the only relations the non-constant symmetric polynomials in y_1, \dots, y_n .

This has a cell decomposition. For a flag $E_\bullet \in \mathbb{F}\ell(n)$ and permutation $w \in S_n$, the set

$$X_w^\circ E_\bullet := \{F_\bullet \in \mathbb{F}\ell(n) \mid \dim(F_a \cap E_b) = \#\{i \leq a \mid w(i) \geq n+1-b\} \forall a, b\},$$

is a topological cell of codimension $\ell(w)$. Writing $[X_w E_\bullet]$ for the cohomology class Poincaré dual to the closure of $X_w^\circ E_\bullet$, these form a basis for the cohomology of $\mathbb{F}\ell(n)$,

$$H^*(\mathbb{F}\ell(n)) = \bigoplus_{w \in S_n} \mathbb{Q}[X_w E_\bullet].$$

By [5, 8, 15] the homomorphism $\psi: \mathbb{Q}[x_1, x_2, \dots] \rightarrow H^*(\mathbb{F}\ell(n))$ that sends x_i to y_i when $i \leq n$ and to 0 for $i > n$, sends the Schubert polynomial \mathfrak{S}_w to the Schubert cycle $[X_w E_\bullet]$ and $\frac{1}{r!} p_r(x_1, \dots, x_k)$ to the tautological class $\text{ch}_r(\mathcal{F}_k^\vee)$. Thus the Murnaghan-Nakayama rule for Schubert polynomials computes the intersection of Schubert cycles with tautological classes.

Given $a_\bullet: 0 < a_1 < \dots < a_s < n$, the partial flag manifold $\mathbb{F}(a_\bullet; n)$ consists of all flags

$$0 \subset F_{a_1} \subset F_{a_2} \subset \dots \subset F_{a_s} \subset \mathbb{C}^n,$$

where $\dim(F_{a_i}) = a_i$. The cohomology of $\mathbb{F}(a_\bullet; n)$ is a subring of $H^*(\mathbb{F}\ell(n))$. It has a basis of Schubert cycles $[X_w E_\bullet]$, where the descent set of w is a subset of $\{a_1, \dots, a_s\}$. Consequently, the Murnaghan-Nakayama rule for Schubert polynomials, when $k \in \{a_1, \dots, a_s\}$, computes the intersection of Schubert cycles in $\mathbb{F}(a_\bullet; n)$ with tautological classes.

4. MURNAGHAN-NAKAYAMA RULE IN THE QUANTUM COHOMOLOGY OF A GRASSMANNIAN

Additively, the quantum cohomology ring $qH^*(\text{Gr}(k, n))$ of the Grassmannian is equal to $H^*(\text{Gr}(k, n))[q]$, where q is an indeterminate of degree n . That is, it has a basis $q^d \sigma_\lambda$ for d a nonnegative integer and $\lambda \leq \square_{k,n}$. The product, $*$, encodes three-point Gromov-Witten invariants [10]. This was the first space whose quantum cohomology was computed [6, 12, 27, 30, 33]. Its ring structure is given by

$$\Lambda_k[q]/\langle h_{n-k+1}, \dots, h_{n-1}, h_n + (-1)^k q \rangle \xrightarrow{\cong} qH^*(\text{Gr}(k, n))$$

where e_a is sent to σ_{1^a} , the a th Chern class of the dual of the tautological bundle. If $q = 0$, we recover the usual cohomology by (2.3). Bertram [6] showed that the image of a Schur polynomial s_λ for $\lambda \leq \square_{k,n}$ is the Schubert cycle σ_λ , just as in ordinary cohomology. Thus $p_r(x_1, \dots, x_k)$ for $r < n$ represents tautological classes as in ordinary cohomology.

Quantum cohomology is also a quotient of Λ_k . Sending e_a to σ_{1^a} as before and $s_{(n-k+1, 1^{k-1})}$ to the quantum parameter q gives an isomorphism

$$(4.1) \quad \Lambda_k / \langle h_{n-k+1}, \dots, h_{n-1} \rangle \xrightarrow{\cong} qH^*(\text{Gr}(k, n)).$$

To see that this surjective map is an isomorphism we consider the identity

$$h_a - e_1 h_{a-1} + e_2 h_{a-2} - \dots + (-1)^a e_a = 0$$

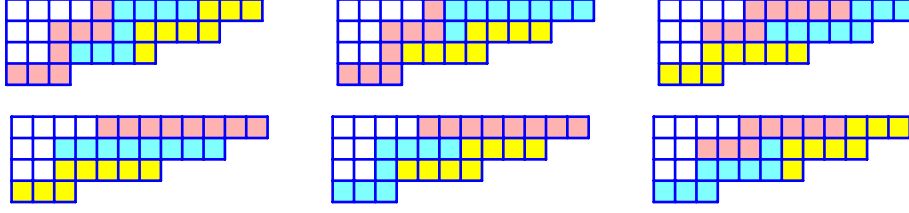
which specializes when $a = n$ to $h_n + (-1)^k s_{(n-k+1, 1^{k-1})} = 0$ since $e_k h_{n-k} = s_{(n-k+1, 1^{k-1})}$ in $\Lambda_k / \langle h_{n-k+1}, \dots, h_{n-1} \rangle$.

In their study of representations of Hecke algebras at n th roots of unity, Goodman and Wenzl [11] defined a (seemingly) different quotient of Λ_k ,

$$(4.2) \quad \Lambda_{k,n} := \Lambda_k / \langle s_\lambda \mid \lambda_1 - \lambda_k = n - k + 1 \rangle,$$

and showed that the images of Schur polynomials s_λ with $\lambda_1 - \lambda_k \leq n - k$ form a basis. They obtained a formula for the corresponding Littlewood-Richardson coefficients with respect to this basis that was equal to a formula obtained by Kac [13, Exer. 13.35] and Walton [31] for fusion coefficients in a Wess-Zumino-Witten conformal field theory. Later, Bertram, Ciocan-Fontanine, and Fulton obtained the same formula for the Littlewood-Richardson coefficients in the quantum cohomology of the Grassmannian [7], showing that $\Lambda_{k,n}$ is isomorphic to $qH^*(\text{Gr}(k, n))$ and that the ideals in (4.1) and (4.2) coincide. Write ψ for the map $\Lambda_k \rightarrow qH^*(\text{Gr}(k, n))$ implicitly defined by either (4.1) or (4.2).

Let λ be a partition with at most k parts. The n -core $\widehat{\lambda}$ of λ is obtained from λ by removing rim hooks of size n until it is not possible to remove any more. The result is independent of choices [17]. When $k = 4$ and $\lambda = (12, 10, 7, 3)$, its 8-core is $\widehat{\lambda} = (4, 2, 2, 0)$, and there are six different ways to remove 8-hooks from λ .



Since $\square_{k,n}$ has $n-1$ diagonals, it is impossible to remove a rim hook of size n from any partition $\lambda \leq \square_{k,n}$. Consequently, all partitions $\lambda \leq \square_{k,n}$ are n -cores.

The image $\psi(s_\lambda)$ of a Schur polynomial $s_\lambda \in \Lambda_k$ in $qH^*(\text{Gr}(k, n))$ was given in [7] by

$$(4.3) \quad s_\lambda \longmapsto \begin{cases} (-1)^{ks - \sum \text{ht}(\rho_i)} q^s \sigma_{\widehat{\lambda}} & \text{if } \widehat{\lambda} \leq \square_{k,n} \\ 0 & \text{otherwise} \end{cases},$$

where we remove s rim hooks ρ_1, \dots, ρ_s of size n from λ to obtain its n -core $\widehat{\lambda}$. Thus $\psi(s_{(12,10,7,3)}) = q^3 s_{(4,2,2)}$, as $k = 4$ and the sum of the heights of the three hooks is even. Similarly, $\psi(s_{(9,8,5,2)}) = 0$, as the 8-core of $(9, 8, 5, 2)$ is $(7, 4, 3, 2)$, and $(7, 4, 3, 2) \not\leq \square_{4,8}$.

Theorem 2. *Let $k, r < n$ be positive integers and $\lambda \leq \square_{k,n}$. Then*

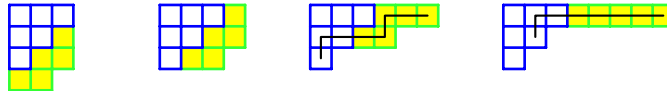
$$(4.4) \quad p_r * \sigma_\lambda = \sum_{\mu} (-1)^{\text{ht}(\mu/\lambda)+1} \sigma_\mu - (-1)^k q \sum_{\nu} (-1)^{\text{ht}(\lambda/\nu)+1} \sigma_\nu,$$

where the first sum is over all $\mu \leq \square_{k,n}$ with μ/λ a rim hook of size r and the second sum is over all $\nu \leq \lambda$ with λ/ν a rim hook of size $n-r$.

Suppose that $k = 4, n = 8$, and $r = 5$, then

$$(4.5) \quad p_5 * \sigma_{(3,2,1)} = \sigma_{(3,3,3,2)} + \sigma_{(4,4,3)} + q\sigma_{(3)} + q\sigma_{(1,1,1)}.$$

To see this, first consider all four ways of adding a rim hook of size 5 to $(3, 2, 1)$,



For the last two, whose first row exceeds $8-4$, we indicate their unique rim hooks of size 8. Removing them gives partitions $\nu \leq \lambda$ with λ/ν a rim hook of size $3 = 8-5$. Applying the formulas for the signs in (4.4) gives the expression (4.5).

Proof. Let $\lambda \leq \square_{k,n}$ and $1 \leq r < n$. The Murnaghan-Nakayama rule in Λ_k is

$$(4.6) \quad p_r \cdot s_\lambda = \sum_{\mu} (-1)^{\text{ht}(\mu/\lambda)+1} s_\mu,$$

the sum over all partitions μ with k parts such that μ/λ is a rim hook of size r . We apply the map ψ to (4.6) to obtain a formula for $p_r * \sigma_\lambda$. Terms of the sum (4.6) indexed by partitions $\mu \leq \square_{k,n}$ contribute to the first, classical, sum in (4.4), as such μ are n -cores. Suppose that $\mu \not\leq \square_{k,n}$ indexes a term in the sum (4.6). Let ν be the n -core of μ . If $\nu \not\leq \square_{k,n}$, then $\psi(s_\mu) = 0$. If $\nu \leq \square_{k,n}$, then we will show (1) that μ/ν is a rim hook of size n , (2) that $\nu \leq \lambda$ with λ/ν is a rim hook of size $n-r$, and (3) that

$$\text{ht}(\lambda/\nu) + \text{ht}(\mu/\lambda) = \text{ht}(\mu/\nu) + 1.$$

These together imply that

$$(4.7) \quad \psi((-1)^{\text{ht}(\mu/\lambda)+1} s_\mu) = q(-1)^{\text{ht}(\mu/\nu)+1+k-\text{ht}(\mu/\nu)} \sigma_\nu = -(-1)^k q(-1)^{\text{ht}(\lambda/\nu)+1} \sigma_\nu.$$

We complete the proof by showing (4) that all partitions $\nu \leq \lambda$ with λ/ν a rim hook of size $n-r$ arise in this way.

A rim hook is determined by either its Southwestern-most or Northeastern-most box. If we have a rim hook of size t in a partition κ with k parts whose Northeastern-most box is the last box in row i of κ , then that rim hook consists of the t consecutive boxes along the rim of κ starting from the end of row i and moving Southwestward. Necessarily, $\kappa_i+k-i \geq r$, as there are only κ_i+k-i diagonals Southwest of this first box.

Suppose that $\mu \not\leq \square_{k,n}$ is obtained from $\lambda \leq \square_{k,n}$ by adding a rim hook of size r . Then the Northeastern-most box of μ/λ is in the first row of μ . Moreover, we either have that $\mu_2 = \lambda_1+1$ or $\mu_2 = \lambda_2$, depending on whether or not the rim hook has height exceeding 1. In either case, it is not possible to remove a rim hook of size n from μ starting from any row other than 1.

Consequently, if we remove a rim hook of size n from μ to obtain a partition ν , we necessarily first remove the rim hook μ/λ , and then a further $n-r$ boxes from λ . This implies that ν satisfies $\nu \leq \lambda$, and so it is an n -core, which implies claims (1) and (2). The rim hook λ/ν has Northeastern-most box in the same row as the Southwestern-most box of μ/λ , which implies claim (3) and (4.7).

For the last claim, as $\square_{k,n}$ contains $n-1$ diagonals, we may add a rim hook of size n to any partition $\kappa \leq \square_{k,n}$ starting in any row $i \leq k$, obtaining a partition $\kappa' \not\leq \square_{k,n}$. Thus if $\nu \leq \lambda$ is any partition with λ/ν a rim hook of size $n-r$, we may extend this rim hook rightward to a rim hook of size n . If μ is the partition obtained, then $\mu \not\leq \square_{k,n}$, μ/λ is a rim hook of size r , μ/ν is a rim hook of size n , and ν is the n -core of μ . \square

Theorem 2 can be extended to multiplication by the image $\psi(p_r)$ for $r \geq n$, as in that case $\psi(p_r) = (-1)^k q \psi(p_{r-n})$, which may be seen using (2.2) and (4.3).

REFERENCES

- [1] J. Bandlow, A. Schilling, and M. Zabrocki, *The Murnaghan-Nakayama rule for k -Schur functions*, J. Combin. Theory Ser. A **118** (2011), no. 5, 1588–1607.
- [2] N. Bergeron and F. Sottile, *Schubert polynomials, the Bruhat order, and the geometry of flag manifolds*, Duke Math. J. **95** (1998), no. 2, 373–423.
- [3] N. Bergeron and F. Sottile, *A monoid for the Grassmannian Bruhat order*, Europ. J. Combin. **20** (1999), no. 3, 197–211.
- [4] N. Bergeron and F. Sottile, *A Pieri-type formula for isotropic flag manifolds*, Trans. AMS **354** (2002), no. 7, 2659–2705.
- [5] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, *Schubert cells and cohomology of the spaces G/P* , Russian Mathematical Surveys **28** (1973), no. 3, 1–26.
- [6] A. Bertram, *Quantum Schubert calculus*, Adv. Math. **128** (1997), no. 2, 289–305.
- [7] A. Bertram, I. Ciocan-Fontanine, and Wm. Fulton, *Quantum multiplication of Schur polynomials*, J. Algebra **219** (1999), 728–746.
- [8] M. Demazure, *Désingularization des variétés de Schubert généralisées*, Ann. Sc. E. N. S. (4) **7** (1974), 53–88.

- [9] S. Fomin and C. Greene, *Noncommutative Schur functions and their applications*, Discrete Math. **193** (1998), no. 1-3, 179–200.
- [10] Wm. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, AMS, 1997, pp. 45–96.
- [11] F. Goodman and H. Wenzl, *Littlewood-Richardson coefficients for Hecke algebras at roots of unity*, Adv. Math. **82** (1990), no. 2, 244–265.
- [12] K. Intriligator, *Fusion residues*, Mod. Phys. Lett. A **6** (1991), 3543–3556.
- [13] V. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., CUP, Cambridge, UK, 1990.
- [14] M. Konvalinka, *Skew quantum Murnaghan-Nakayama rule*, J. Alg. Comb. **35** (2012), no. 4, 519–545.
- [15] A. Lascoux and M.-P. Schützenberger, *Polynômes de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), no. 13, 447–450.
- [16] I.G. Macdonald, *Notes on Schubert polynomials*, Laboratoire de combinatoire et d’informatique mathématique (LACIM), Univ. du Québec à Montréal, Montréal, 1991.
- [17] I.G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995.
- [18] P. McNamara, *A Pieri rule for skew shapes*, Talk at FPSAC 2010, www.facstaff.bucknell.edu/pm040/Slides/fpsac10.pdf.
- [19] D. Monk, *The geometry of flag manifolds*, Proc. London Math. Soc. **9** (1959), 253–286.
- [20] A. Morrison, *A Murnaghan-Nakayama rule for Schubert polynomials*, DMTCS Proceedings, 26th Intl. Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2014), 2014.
- [21] F. Murnaghan, *The characters of the symmetric group*, Amer. J. Math. **59** (1937), no. 4, 739–753.
- [22] T. Nakayama, *On some modular properties of irreducible representations of a symmetric group, I*, Jap. J. Math **18** (1941), 89–108.
- [23] T. Nakayama, *On some modular properties of irreducible representations of a symmetric group, II*, Jap. J. Math **17** (1941), no. 165, 411.
- [24] M. Pieri, *Sul problema degli spazi secanti*, Rend. Ist. Lombardo **26** (1893), 534–546.
- [25] D. Ross, *The loop Murnaghan-Nakayama rule*, J. Algebraic Combin. **39** (2014), no. 1, 3–15.
- [26] D. Ross and Z. Zong, *The gerby Gopakumar-Mariño-Vafa formula*, Geom. Topol. **17** (2013), no. 5, 2935–2976.
- [27] B. Siebert and G. Tian, *On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator*, Asian J. Math. **1** (1997), 679–695.
- [28] F. Sottile, *Pieri’s formula for flag manifolds and Schubert polynomials*, Ann. Inst. Fourier (Grenoble) **46** (1996), no. 1, 89–110.
- [29] V. Tewari, *A Murnaghan-Nakayama rule for noncommutative Schur functions*, arXiv:1403.0607.
- [30] C. Vafa, *Topological mirrors and quantum rings*, Essays on Mirror Manifolds, International Press, 1992, ed. by S.-T. Yau, pp. 96–119.
- [31] M. Walton, *Fusion rules in Weiss-Zumino-Witten models*, Nuclear Phys. B **340** (1990), no. 2-3, 777–790.
- [32] M. Wildon, *A short proof of a plethystic Murnaghan-Nakayama rule*, arXiv:1408.3554.
- [33] E. Witten, *The Verlinde algebra and the cohomology of the Grassmannian*, Geometry, Topology, and Physics, Lecture Notes in Geometric Topology, vol. IV, Int. Press, 1995, pp. 357–422.

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