A New Approach to Hilbert’s Theorem on Ternary Quartics

Une nouvelle approche du théorème de Hilbert sur les quartiques ternaires

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Abstract

Hilbert proved that a non-negative real quartic form \( f(x, y, z) \) is the sum of three squares of quadratic forms. We give a new proof which shows that if the plane curve \( Q \) defined by \( f \) is smooth, then \( f \) has exactly 8 such representations, up to equivalence. They correspond to those real 2-torsion points of the Jacobian of \( Q \) which are not represented by a conjugation-invariant divisor on \( Q \).

Résumé

Hilbert a démontré qu’une forme réelle non négative \( f(x, y, z) \) de degré 4 est la somme de trois carrés de formes quadratiques. Nous donnons une nouvelle démonstration qui montre que si la courbe plane \( Q \) définie par \( f \) est non singulière, alors \( f \) a exactement 8 telles représentations, à équivalence près. Elles correspondent aux points de 2-torsion du jacobien de \( Q \) qui ne sont pas représentés par un diviseur de \( Q \) invariant par conjugaison.

1. Introduction

A ternary quartic form is a homogeneous polynomial \( f(x, y, z) \) of degree 4 in three variables. If \( f \) has real coefficients, then \( f \) is non-negative if \( f(x, y, z) \geq 0 \) for all real \( x, y, z \). Hilbert [5] showed that every non-negative real ternary quartic form is a sum of three squares of quadratic forms. His proof (see [8], [9] for modern expositions) was non-constructive: The map

\[
\pi: (p, q, r) \mapsto p^2 + q^2 + r^2
\]
from triples of real quadratic forms to non-negative quartic forms is surjective, as it is both open and closed when restricted to the preimage of the (dense) connected set of non-negative quartic forms which define a smooth complex plane curve. An elementary and constructive approach to Hilbert’s theorem was recently begun by Pfister [6].

A quadratic representation of a complex ternary quartic form \( f = f(x, y, z) \) is an expression

\[
f = p^2 + q^2 + r^2
\]

where \( p, q, r \) are complex quadratic forms. A representation \( f = (p')^2 + (q')^2 + (r')^2 \) is equivalent to this if \( p, q, r \) and \( p', q', r' \) have the same linear span in the space of quadratic forms.

Powers and Reznick [7] investigated quadratic representations computationally, using the Gram matrix method of [1]. In several examples of non-negative real ternary quartics, they always found 63 inequivalent representations as a sum of three squares of complex quadratic forms; 15 of these were sums or differences of squares of real forms. We explain these numbers, in particular the number 15, and show that precisely 8 of the 15 are sums of squares.

If the complex plane curve \( Q \) defined by \( f = 0 \) is smooth, it has genus 3, and so the Jacobian \( J \) of \( Q \) has \( 2^6 - 1 = 63 \) non-zero 2-torsion points. Coble [2, Chap 1,§14] showed that these are in one-to-one correspondence with equivalence classes of quadratic representations of \( f \). If \( f \) is real, then \( Q \) and \( J \) are defined over \( \mathbb{R} \). The non-zero 2-torsion points of \( J(\mathbb{R}) \) correspond to signed quadratic representations \( f = \pm p_1^2 \pm p_2^2 \pm p_3^2 \), where \( p_i \) are real quadratic forms. If \( f \) is also non-negative, the real Lie group \( J(\mathbb{R}) \) has two connected components, and hence has \( 2^4 - 1 = 15 \) non-zero 2-torsion points. We use Galois cohomology to determine which 2-torsion points give rise to sum of squares representations over \( \mathbb{R} \).

**Theorem 1** Suppose that \( f(x, y, z) \) is a non-negative real quartic form which defines a smooth plane curve \( Q \). Then the inequivalent representations of \( f \) as a sum of three squares are in one-to-one correspondence with the eight 2-torsion points in the non-identity component of \( J(\mathbb{R}) \), where \( J \) is the Jacobian of \( Q \).

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## 2. Quadratic representations of smooth ternary quartics

Let \( f(x, y, z) \) be an irreducible quartic form over \( \mathbb{C} \), and let \( Q \) be the curve \( f = 0 \) in the complex projective plane. Assume that \( Q \) is smooth. The Picard group \( \text{Pic}(Q) \) of \( Q \) is the group of Weil divisors on \( Q \), modulo divisors of rational functions. Let \( J \) be the Jacobian of \( Q \), so that \( J \) is the identity component of \( \text{Pic}(Q) \). The following proposition is due to Coble [2, Chap 1,§14].

**Proposition 1** The non-trivial 2-torsion points of \( J \) are in one-to-one correspondence with the equivalence classes of quadratic representations of \( f \).

**Proof.** Given a quadratic representation \( (1) \), consider the map

\[
\varphi : \mathbb{P}^2 \to \mathbb{P}^2, \quad x \mapsto (p(x) : q(x) : r(x)).
\]

The image of \( Q \) under \( \varphi \) is the conic \( C \) defined by the equation \( y_0^2 + y_1^2 + y_2^2 = 0 \). Let \( y \) be any point in \( C \), then \( \varphi^*(y) \) is an effective divisor of degree 4 that is not the divisor of a linear form. Indeed, after a linear change of coordinates we can assume \( y = (0 : 1 : i) \). A linear form vanishing on \( \varphi^*(y) \) would divide each conic \( \alpha p + \beta (q + ir) \) through \( \varphi^*(y) \), and thus would divide

\[
f = p^2 + (q + ir)(q - ir),
\]

contradicting the irreducibility of \( f \).

Fix a linear form \( \ell \), then \( L := \text{div}(\ell) \) is an effective divisor of degree 4 on \( Q \). Let \( \zeta = [\varphi^*(y) - L] \). Since \( 2y \) is the divisor of a linear form (the tangent line to \( C \) at \( y \), \( \varphi^*(2y) \) is the divisor on \( Q \) of a quadratic
form. Thus $2\zeta = 0$. Moreover, $\zeta \neq 0$ as $\varphi^*(y)$ is not the divisor of a linear form. The 2-torsion point $\zeta$ of $J$ depends only upon the map $\varphi$.

Conversely, suppose that $\zeta \in J(C)$ is a non-zero 2-torsion point. Let $D \neq D'$ be effective divisors which represent the class $\zeta + [L]$ in Pic($Q$). As $Q$ has genus 3, the Riemann-Roch Theorem implies that there is a pencil of such divisors. Then $2D$, $2D'$ and $D + D'$ are effective divisors of degree 8, and are all linearly equivalent to $2L$, the divisor of a conic. Again from the Riemann-Roch Theorem it follows that there are quadratic forms $q_0$, $q_1$ and $q_2$ such that

$$\text{div}(q_0) = 2D, \quad \text{div}(q_1) = 2D' \quad \text{and} \quad \text{div}(q_2) = D + D'.$$

Therefore, the rational function $g := q_0q_1/q_2^2$ on $Q$ is constant. Scaling $q_1$ and $q_2$ appropriately, we may assume that $g \equiv 1$ on $Q$ and also that $f = q_0q_1 - q_2^2$. Diagonalizing the quadratic form $q_0q_1 - q_2^2$ gives a quadratic representation for $f$. This defines the inverse of the previous map. □

3. Quadratic representations of real quartics

Suppose now that $f$ is a non-negative real quartic form defining a smooth real plane curve $Q$ with complexification $Q_C = Q \otimes_{\mathbb{R}} \mathbb{C}$. The elements of Pic($Q$) can be identified with those divisor classes in Pic($Q_C$) that are represented by a conjugation-invariant divisor. Let $J$ be the Jacobian of $Q$.

If $\zeta \in J(C)$ is the 2-torsion point corresponding to a signed quadratic representation

$$f = \pm p^2 \pm q^2 \pm r^2$$

consisting of real polynomials $p$, $q$, $r$, then $\zeta = \overline{\zeta}$, i.e., $\zeta \in J(\mathbb{R})$.

Conversely, let $0 \neq \zeta \in J(\mathbb{R})$ with $2\zeta = 0$, and let $L$ be the divisor on $Q$ of a linear form $\ell$. We can choose an effective divisor $D \neq \overline{D}$ on $Q_C$ representing the class $\zeta + [L]$. Then $2D$, $2\overline{D}$ and $D + \overline{D}$ are each equivalent to $2L$. Let $r$ be a real quadratic form with divisor $D + \overline{D}$, and let $g$ be a complex quadratic form with divisor $2D$ (both divisors taken on $Q_C$).

Since $D \sim \overline{D}$, there is a rational function $h$ on $Q_C$ with $\text{div}(h) = \overline{D} - D$. Let $c = h\overline{h}$, a nonzero real constant on $Q$. Since $\text{div}(r) = \text{div}(g) + \text{div}(h)$, there is a complex number $\alpha \neq 0$ with $\frac{r}{g} = \alpha h$ on $Q$, which implies that

$$c|q|^2 = \frac{r}{g} \cdot \frac{\overline{r}}{\overline{g}} = \frac{r^2}{p^2 + q^2}$$

on $Q$, where $p$ and $q$ are the real and imaginary parts of $g = p + iq$. So the quartic form

$$u := r^2 - c|q|^2(p^2 + q^2)$$

vanishes identically on $Q$. Since $u \neq 0$, $f$ is a constant multiple of $u$. If $c > 0$, we get a signed quadratic representation of $f$, with both signs occurring. If $c < 0$, $f$ must be a positive multiple of $u$ since $f$ is non-negative, and we get a representation of $f$ as a sum of three squares of real forms.

We now calculate the sign of $c$. For this we use the well-known exact sequence

$$0 \to \text{Pic}(Q) \to \text{Pic}(Q_C)^G \to \text{Br}(\mathbb{R}) \to \text{Br}(Q).$$

It arises from the Hochschild-Serre spectral sequence for étale cohomology with coefficients $\mathbb{G}_m$. Here $G = \text{Gal}(\mathbb{C}/\mathbb{R})$ acts on Pic($Q_C$) by conjugation, and Pic($Q_C)^G$ is the group of $G$-invariant divisor classes. Moreover, Br($\mathbb{R}$) is the Brauer group of $\mathbb{R}$, which is of order 2, and Br($Q$), the Brauer group of $Q$, can be identified with the subgroup of Br($\mathbb{R}$) consisting of all Brauer classes which are everywhere unramified. The map Br($\mathbb{R}$) → Br($Q$) is the restriction map.

It is easy to see that $c < 0$ if and only if $\partial(\zeta)$ is the non-trivial class in Br($\mathbb{R}$).

By a classical theorem of Witt [12], every non-negative rational function on a smooth projective curve over $\mathbb{R}$ is a sum of two squares of rational functions. Since $Q$ is smooth and $f$ is non-negative, this forces
$Q(\mathbb{R}) = \emptyset$. Hence $-1$ is a sum of two squares in $\mathbb{R}(Q)$. This means $(-1, -1) = 0$ in $\text{Br}(Q)$, and hence the map $\partial$ is surjective.

Since the genus of $Q$ is odd (equal to 3), it follows from a classical theorem of Weichold [11,3] that all classes in $\text{Pic}(Q_C)^0$ have even degree, and the real Lie group $J(\mathbb{R})^0$ has exactly two connected components. This implies that the sequence

$$0 \rightarrow J(\mathbb{R})^0 \rightarrow J(\mathbb{R}) \xrightarrow{\partial} \text{Br}(\mathbb{R}) \rightarrow 0$$

is (split) exact. Since $J(\mathbb{R})^0 \cong (S^1)^3$ as a real Lie group, there exist $2^4 - 1 = 15$ non-zero 2-torsion classes in $J(\mathbb{R})$. The 8 that do not lie in $J(\mathbb{R})^0$, or equivalently, which cannot be represented by a conjugation-invariant divisor on $Q_C$, are precisely those that give rise to sums of squares representations of $f$. This completes the proof of Theorem 1.

We close with a few remarks about the singular case. Wall [10] studies quadratic representations of (possibly singular) complex ternary quartic forms $f$. If $f$ is irreducible, the non-trivial 2-torsion points on the generalized Jacobian of the curve $Q = \{f = 0\}$ again give equivalence classes of quadratic representations of $f$. These representations are special in that they have no basepoints.

Quadratic representations with a given base locus $B \neq \emptyset$ are in one-to-one correspondence with all 2-torsion points on the Jacobian of a curve $Q$, which is the image of $Q$ under the complete linear series of quadrics through $B$. By classifying all possibilities for $B$ one arrives at the number of inequivalent quadratic representations of $f$. If the form $f$ is real and non-negative, this classification, together with arguments from Galois cohomology, gives all inequivalent representations of $f$ as a sum of squares. If $f$ is reducible, different methods can be applied to complete the picture. This complete analysis will appear in an unabridged version.

References