NUMERICAL REAL ALGEBRAIC GEOMETRY

Bernstein's Theorem: Exercises

(1) Solve the following binomial systems

(a) $x^5y^3 - 2xy^{-1} \;=\; 5xy - x^3y^4 \;=\; 0.$

(b) $x^{11}y - 3x^3y^{-5} \;=\; xy^{13} - 7x^3y^4 \;=\; 0.$

(c) $xyz^2 - x^2y^{-1} \;=\; x^3y^3 - 2x^{-1}z^3 \;=\; y^3 - xz^3 \;=\; 0.$

(d) $x^3w - 5y^2z^{-3} \;=\; y^3w^5 - x^7z \;=\; xyz - 7y^5w^9 \;=\; y^3 - x^4yz^4w^2 \;=\; 0.$

(2) Determine the Newton polytope of each polynomial, and the mixed volume of each polynomial systyem. Check the conclusion of Bernstein's Theorem using a computer algebra system such as Singular, Macualay2, or CoCoA.

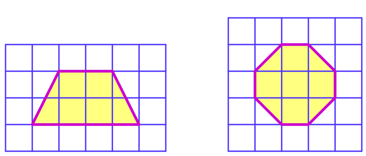
(a)
$$1 + 2x + 3y + 4xy \;=\; 1 - 2xy + 3x^2y - 5xy^2 \;=\; 0$$

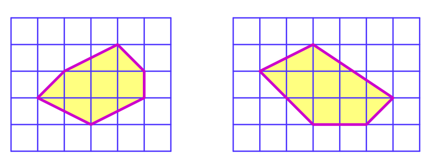
(b)
$$\begin{aligned} 1 + 2x + 3y - 5xy + 7x^2y^2 &= 0 \\ 1 - 2xy + 4x^2y + 8xy^2 - 16x^3y + 32xy^3 - 64x^2y^2 &= 0 \end{aligned}$$

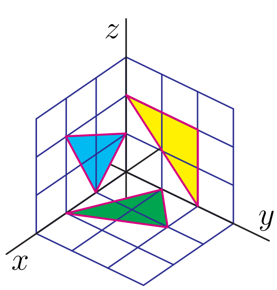
(c)
$$\begin{aligned} 2 + 5xy - x^2y - 6xy^2 + 4xy^3 &= 0 \\ 2x - y - 2y^2 - xy^2 + 2x^2y + x^2 - 5xy &= 0 \end{aligned}$$

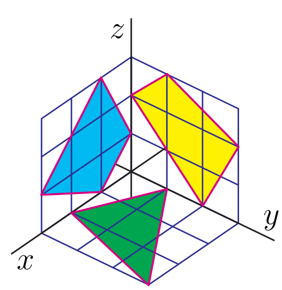
(d)
$$\begin{aligned} 1 + x + y + z + xy + xz + yz + xyz &= 0 \\ xy + 2xyz + 3xyz^2 + 5xz + 7xy^2z + 11yz + 13x^2yz &= 0 \\ 4 - x^2y + 2x^2z - xz^2 + 2yz^2 - y^2z + 2y^2x - 8xyz &= 0 \end{aligned}$$

(3) Determine the mixed volumes of each of the lists of polytopes given below and verify Bernstein's Theorem by solving a randomly generated polynomial system with the given Newton polytopes.

(a) 

(b) 

(c) 

(d) 

(4) Show that Newton polytopes are additive under polynomial multiplication,

$$\mathrm{New}(f \cdot g) \;=\; \mathrm{New}(f) + \mathrm{New}(g)\,.$$

(5) Prove the following formula for the mixed area of two polygons P and Q.

$$\mathrm{MV}(P, Q) \;=\; \mathrm{vol}(P+Q) \;-\; \mathrm{vol}(P) \;-\; \mathrm{vol}(Q)\,.$$

(6) Find integer lifting functions and compute the corresponding mixed subdivision of the lists of polytopes in Exercise 2.

(7) Kushnirenko's Theorem is the special case of Bernstein's Theorem when all the Newton polytopes are equal. Formulate a statement of Kushnirenko's Theorem and deduce it from Bernstein's Theorem.

(8) For each $i = 1, \ldots, k$ let $x^{(i)} := (x_1^{(i)}, \ldots, x_{n_i}^{(i)})$ be a list of n_i variables. A polynomial $f(x^{(1)}, \ldots, x^{(k)})$ has *multidegree* $\mathbf{d} = (d_1, \ldots, d_k)$ if it has degree d_i in the variables $x^{(i)}$, for each $i = 1, \ldots, k$. Use Kushnirenko's Theorem to deduce the bound for the number of common zeroes to a system of $n = n_1 + \cdots + n_k$ equations that has multidegree $\mathbf{d}$.