Higher Convexity for Complements of Tropical Varieties

Geometry and topology of (co)amoebas

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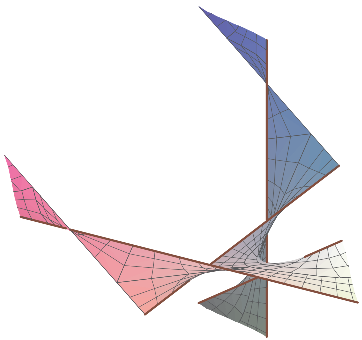
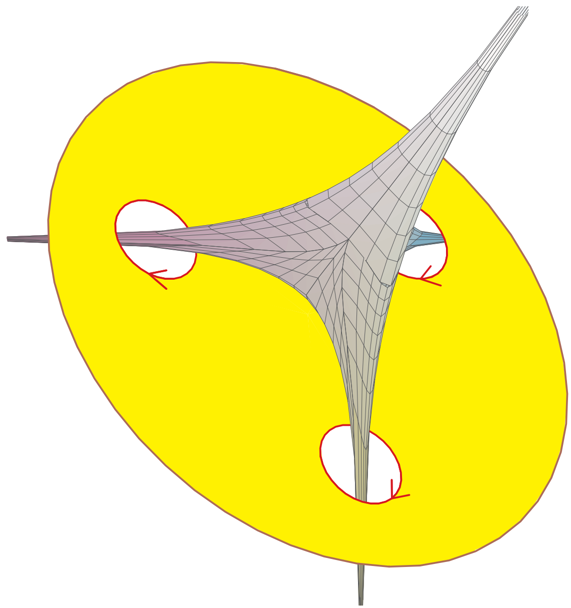


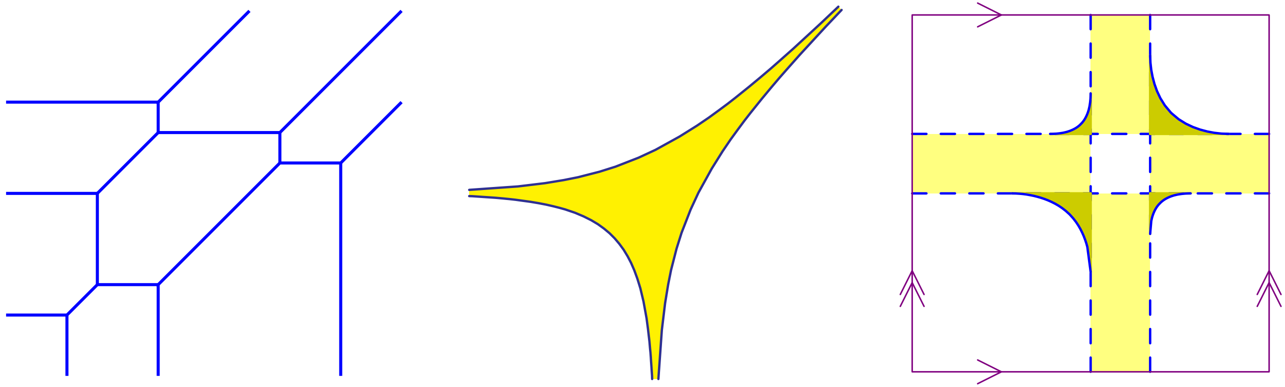


Joint work with Mounir Nisse

Convexity of Hypersurface Complements

Complements of hypersurface objects are unions of convex sets

Tropical Amoeba Coamoeba



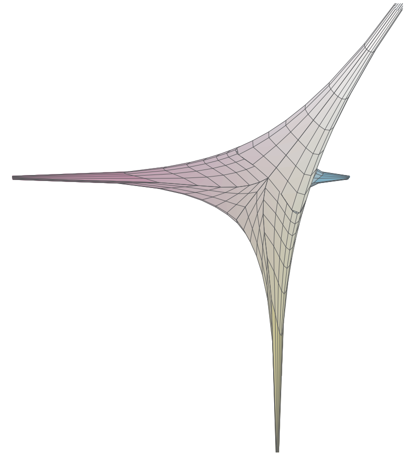
A tropical hypersurface is the ridge set of a polyhedral decomposition of $\mathbb{R}^n$

For amoebas, it is by logarithmic convexity of domains of convergence of power series for $1/f$

Nisse showed this for coamoebas, where we must lift to universal covers

Higher Convexity

In 2004 Henriques considered complements of general amoebas, developing a generalization of convexity for them



Definition An open subset $X \subset \mathbb{R}^n$ is *k-convex* if, for every $(k+1)$-plane L, the natural map $H_k(X \cap L) \to H_k(X)$ is an injection

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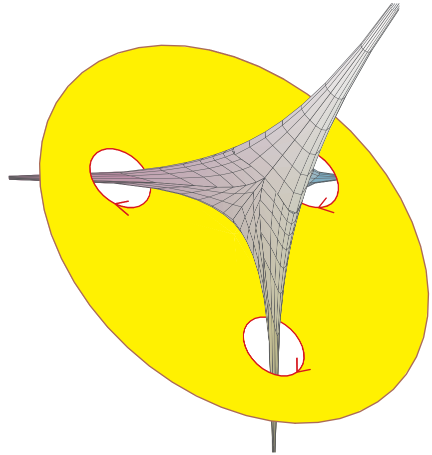

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0-convex and connected is ordinary convexity

Henriques implicitly conjectured that if $V \subset (\mathbb{C}^\times)^n$ has pure codimension $k+1$, then the complement of its amoeba is k-convex

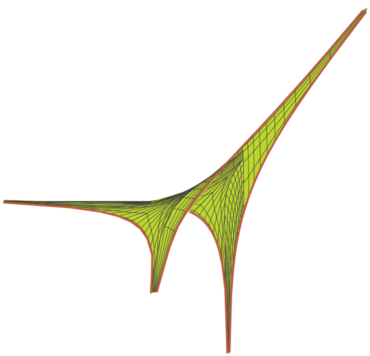
I will focus on what is known, and a little bit why, about this *higher convexity*

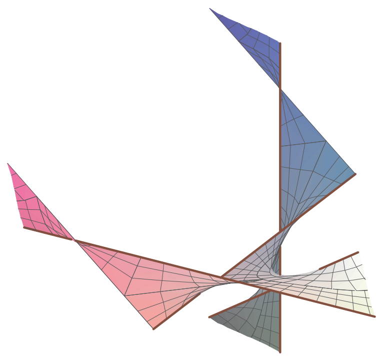
Some known results

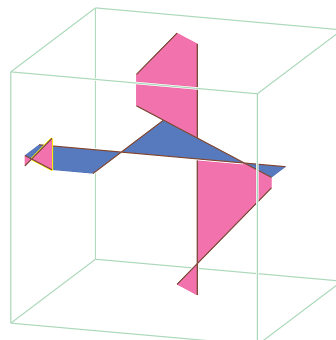
Henriques If X is the complement of the amoeba of a variety of codimension $k+1$, then for any affine $(k+1)$-plane L the map $H_k(X \cap L) \to H_k(X)$ sends no positive cycle to zero. Call this *weakly k-convex*

Bushueva-Tsikh The complement of the amoeba of a variety that is a complete intersection of codimension $k+1$ is k-convex

Nisse-S. The complement of the coamoeba of a variety of codimension $k+1$ is k-convex, and the same for the nonarchimedian coamoeba







Higher Convexity of Tropical Varieties

With Nisse, we offer three theorems:

Theorem *The complement of a tropical curve in* $\mathbb{R}^n$ *is* $(n-2)$ *convex*

Work now over the complex Puiseux field $\mathbb{C}\{\{t\}\} := \mathbb{C}\{\{t^{\mathbb{Q}}\}\}$

Theorem *The complement of the nonarchimedian amoeba of a variety of codimension* $k+1$ *is weakly* k*-convex*

Theorem *The complement of the nonarchimedian amoeba of a variety that is a complete intersection of codimension* $k+1$ *is* k*-convex*

We conjecture that the complement of any tropical variety
(in any sense) of codimension $k+1$ is k-convex

For example, our result on curves is via a direct argument, using only a weak consequence of balancing

Jonsson's Limit Theorem

Let $\mathcal{V} \subset \mathbb{C}^\times \times (\mathbb{C}^\times)^n$ be a family of subvarieties with base $\mathbb{C}^\times$. Write $\mathcal{A}_t$ for the amoeba of the fiber over $t \in \mathbb{C}^\times$ and $\mathcal{T}$ for the nonarchimedean amoeba of the variety in $(\mathbb{C}\{\{t\}\}^\times)^n$ obtained by base change from $\mathbb{C}[t, t^{-1}]$ to $\mathbb{C}\{\{t\}\}$

Theorem [Jonsson] $\lim_{t\to 0} \frac{-1}{\log|t|}\mathcal{A}_t \;=\; \mathcal{T}$, *(in the sense of Kuratowski)*

Starting with a variety $\mathcal{V}$ over $\mathbb{C}\{\{t\}\}$, by Katz there is a variety over a finite extension of $\mathbb{C}(t)$ with the same nonarchimedean amoeba and Hilbert series. After a base change and a scaling, get a variety over $\mathbb{C}(t)$

We actually show that the complement of a polyhedral complex of pure codimension $k{+}1$ is k-convex, if it is the Kuratowski limit of sets whose complements are k-convex.

Gromov's Higher Convexity

Let $X \subset \mathbb{R}^n$ be open with ∂X a manifold, M

Much earlier (1991) Gromov made the following definition:
X is k-convex if at every point m of M, at most k principal curvatures of M are negative (this is with respect to the inward normal to M)

Gromov also proved:

- If X and Y are k-convex, then $X \cap Y$ is k-convex
- If X is k-convex and L an affine $(k+1)$-plane, then $H_k(X \cap L) \to H_k(X)$ is injective

  $\rightsquigarrow$ This implies Henriques' k-convexity

Future Directions

- Adopt Gromov's ideas to show that higher convexity is a local property

  $\rightsquigarrow$ May imply k-convexity for local complete intersections

- Develop methods to prove k-convexity for special fibers of families of varieties

- Prove k-convexity for tropical varieties directly

- Prove Henriques' conjecture for amoebas

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