The Ubiquity of Determinantal Equations

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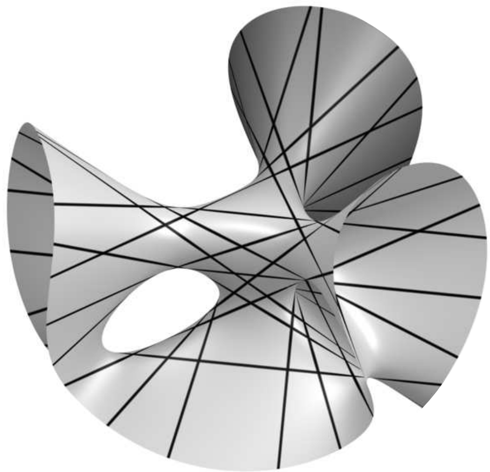


Work with Hauenstein, Hein, Huber, Leykin, Martín del Campo, Sturmfels, Vakil, and Verschelde.

Rank Conditions

All meaningful equations in algebraic geometry arise as rank conditions on parameterized matrices.

This is a matter of faith among algebraic geometers. Let $\phi\colon F_b \to E_a$ be a map of vector bundles of ranks a and b over a variety X.

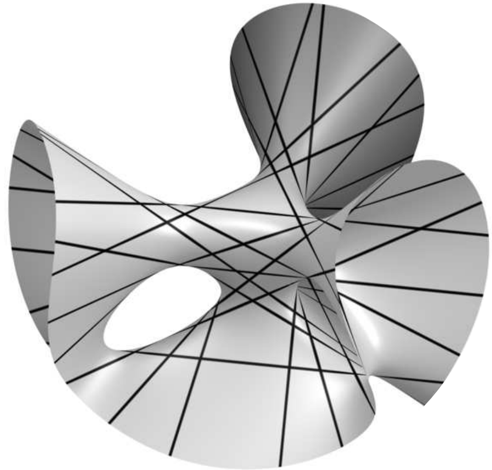
$$\begin{tikzcd} F_b \arrow[r, "\phi"] \arrow[dr] & E_a \arrow[d] \\ & X \end{tikzcd}$$

Set $Z_k\phi := \{x \in X \mid \mathrm{rk}\,\phi(x) \leq k\}$.

Many geometric loci are expressed in terms of the sets $Z_k\phi$ where the map ϕ drops rank. In local coordinates, ϕ is a matrix of functions on X, and $Z_k\phi$ is given by minors of that matrix.

Example: Suppose that $b \leq a$ and $F = \mathbb{C}^b$. Then the *Chern class* $c_b(E)$ is the locus where $a+1-b$ sections of E become dependent.

27 Lines on a Cubic



The Grassmannian $\mathrm{Gr}(2,4)$ is the space of lines ℓ in (projective) 3-space $\mathbb{P}^3$.

A cubic form $C \in \mathrm{Sym}_3(\mathbb{C}^4) \simeq \mathbb{C}^{20}$ defines a cubic surface $\mathcal{V}(C)$ in $\mathbb{P}^3$.

A line ℓ lies in the surface if and only if $C|_\ell$ is identically zero.

Let E be the bundle over $\mathrm{Gr}(2,4)$ whose fiber over a line ℓ is the 4-dimensional space of cubic forms on ℓ. Restriction defines a surjective map $\phi\colon \mathbb{C}^{20} \twoheadrightarrow E$. A cubic form $C \in \mathbb{C}^{20}$ gives a section of E, which vanishes exactly at the lines ℓ that lie in $\mathcal{V}(C)$.

Standard numerical calculations show that $27 = \deg(c_4(E))$, and this *also gives the equations on ℓ to lie in $\mathcal{V}(C)$*.

Wilder, But Comfortable

This is just the beginning of the story (from the 19th and 20th c.) from the perspective of algebraic geometry, whose contours we sketch.

A bundle map $F \xrightarrow{\phi} E$ over a space X corresponds to a section of a Grassmann bundle over X. In local coordinates, this is a map $\Phi\colon X \to \mathrm{Gr}(b, \mathbb{C}^n)$ to a Grassmann variety.

The geometric loci $(Z_k\phi)$ are pullbacks of distinguished *Schubert varieties* on $\mathrm{Gr}(b, \mathbb{C}^n)$. This refines the construction of Chern classes as pullbacks of cohomology classes along maps to Grassmannians classifying bundles.

In practical terms, these sets are defined in terms of rank conditions on matrices $M_{n\times m}(x) = \Phi(x)$ of regular functions on X,

$$Z_k\phi \;=\; \{x \in X \mid \mathrm{rk}\, M_{n\times m}(x) \leq k\}\,.$$

(The general case has this form, but many more equations/rank conditions.)

Dialing Back: The Actual Equations

The basic (universal) case is when X is the Grassmannian so that we have equations of the form

$$\mathrm{rk}[M \mid F^{(i)}] \; < \; k_i \qquad i = 1, \ldots, s\,,$$

where M is a $n \times b$ matrix with entries 0, 1, and indeterminates, and each $F^{(i)}$ is a $n \times c_i$ constant matrix of rank c_i.

Each rank condition is minimally given by some number d_i of linear combinations of maximal minors of M and $d_1 + \cdots + d_s >$ #indeterminates.

This leads to many equations of moderate/high degree that do not form a *square system* (# equations = # indeterminates).

A first goal is to develop tools for systems of this form coming from Grassmannians and more general flag manifolds, and then apply the lessons learned to the more general rank conditions on matrices of functions.

Approaches

There are several methods to solve

$$\mathrm{rk}[M \mid F^{(i)}] \;<\; k_i \qquad i = 1, \ldots, s\,.$$

(1) Just do it. Symbolic are fine with this formulation.
$\longrightarrow$ Standard numerical algorithms do not always perform well.

(2) Exploit geometry of Schubert varieties in homotopy algorithms.
(W/ Huber, Sturmfels, Vakil, Verschelde, Leykin, Martín del Campo.)
$\longrightarrow$ Mathematically beautiful, but algorithmically complicated.
Work in progress. Performs spectacularly for some classes, and is not yet an improvement for others.

(3) Use standard methods (regeneration) adapted to Schubert geometry.
$\longrightarrow$ Promising, but not yet tested.

More ....

To solve

$$\mathrm{rk}[M \mid F^{(i)}] \;<\; k_i \qquad i = 1, \ldots, s\,.$$

(4) Use monodromy. (Easy to find a system given a solution.)
$\longrightarrow$ Promising, but not yet tested.

(5) Solve a different, but equivalent system.

Recall: An $m \times n$ matrix A $(m \geq n)$ has rank $< n$ if and only if
(a) All $\binom{m}{n}$ maximal minors of A vanish *or*
(b) $Ax = 0$ has a solution $0 \neq x \in \mathbb{C}^n$.

The second approach add variables, and replaces many minors by exactly the right number of bilinear equations. The resulting square system has many numerical advantages.

$\longrightarrow$ This is the idea behind Lagrange's method of multipliers.

Generalize $Ax = 0$

$$(*) \qquad \mathrm{rk}[M \mid F^{(i)}] \ < \ k_i \qquad i = 1, \ldots, s\,.$$

Building on work with Hauenstein, with Hein, we extended this idea to give an efficient square system of bilinear equations equivalent to $(*)$, as well as a more general square formulation for all Schubert varieties on Grassmannians and flag manifolds. (type A for lie aficionados).

Besides the mentioned work in progress, it remains an open problem to give a square reformulation when the bundles and maps have some symmetry. (Lie types B, C, D, etc.)

$\longrightarrow$ Likely no square reformulation, but may have a method for certification.