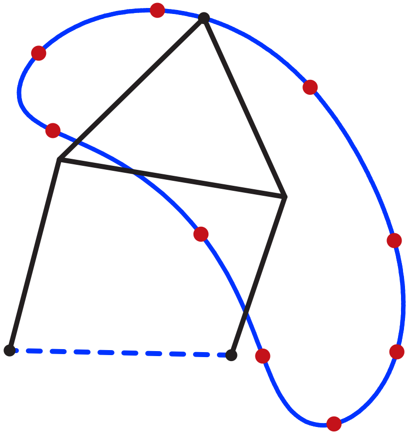
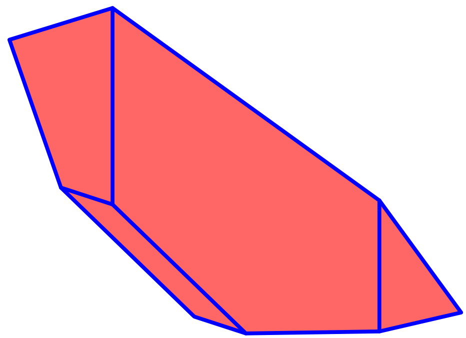
Newton-Okounkov Bodies and Khovanskii Bases for Applications

SIAM Minisymposium on Applications of Newton-Okounkov Bodies

31 July 2017

$\#$  $\stackrel{?}{=}$ 

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Newton-Okounkov Bodies

$\longrightarrow$ Newton-Okounkov bodies significantly generalize and extend the theory and uses of Newton polyhedra from toric varieties.

A Newton-Okounkov body is a convex body $NO(L) \subset \mathbb{R}^n$ associated to a linear series L on a variety X ($L \subset \mathbb{C}(X)$ is finite-dimensional) and a valuation $\nu\colon \mathbb{C}(X)^* \to \mathbb{Z}^n$ ($\dim X = n$). While $NO(L)$ depends upon the valuation, its volume does not.

• They appeared in work of Okounkov in 1998 on multiplicities in representation theory.

• c. 2007 Kaveh-Khovanskii and Lazarsfeld-Mustaţa independently introduced Newton-Okounkov bodies. As I understand it, they had different perspectives/applications in mind.

(Simplified) Definition

Suppose that X is rational, and $L \subset \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ is a finite-dimensional vector space of functions on X.

Define $R(L) := \mathbb{C} \oplus tL \oplus t^2L^2 \oplus t^3L^3 \oplus \cdots$
where $L^d :=$ span$\{f_1 \cdots f_d \mid f_i \in L\}$.

Pick a term order $\prec$ on $\mathbb{C}[x^{\pm}]$. Define $\nu \colon R(L) \to \mathbb{N} \oplus \mathbb{Z}^n$ by

$$t^dL^d \ \ni \ t^dg \ \longmapsto \ (d, \mathrm{in}_{\prec}(g))\,.$$

Then $\Gamma(L) := \nu(R)$ is a subsemigroup of $\mathbb{N} \oplus \mathbb{Z}^n$.

The *Newton-Okounkov body* associated to L is the slice

$$NO(L) \ := \ \mathbb{R}_{\geq 0}\Gamma(L) \cap \{1\} \times \mathbb{R}^n\,.$$

Tomorrow, Kiumars Kaveh will give a more honest development.

Two Results

Set $X(L) := \mathrm{proj}(R(L))$, the closure of image of X in $\mathbb{P}(L^*)$. (Assume this has dimension n.)

Theorem. (O & KK & LM) $n!\,\mathrm{Vol}(NO(L)) = \deg X(L)$.

Corollary. (O & KK & LM) If $(f_1, \ldots, f_n) \in L^{\oplus n}$ is general, then $f_1 = \cdots = f_n = 0$ has $n!\,\mathrm{Vol}(NO(L))$ isolated solutions in X.

Theorem. (Dave Anderson) When $\Gamma(L)$ is finitely generated, there is a flat embedded degeneration of $X(L)$ into the toric variety $\mathrm{proj}(\mathbb{C}[\Gamma(L)])$.

When $\Gamma(L)$ is finitely generated, a finite set $\mathcal{B} \subset R(L)$ whose image generates $\Gamma(L)$ is a *Khovanskii basis* of $R(L)$.

Importance of Newton Okounkov Bodies

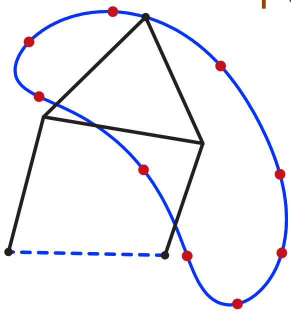
Pure Mathematics: A broad impact of this theory has been the expansion of combinatorial methods in algebraic geometry.
Newton Okounkov bodies were a major theme in the Fall 2016 Fields Institute Programme on Combinatorial Algebraic Geometry.

The other talks in this minisymposium highlight some of this, particularly aspects from representation theory and computational algebraic geometry.

Three talks, by Bossinger, Abe, and Corey, are about toric degenerations of algebraic varieties, and relations to Newton Okounkov bodies, and the talk by Manon is about the existence of Khovanskii bases, in the language of tropical geometry.

Applications of Mathematics: I will quickly sketch two ways I think this theory may have importance for applications of algebraic geometry.

Root Counting (by example)



A four bar mechanism is a plane quadrilateral with a triangle mounted opposite a fixed side. The six 'bars' have fixed lengths, but rotate freely at their joints.
The apex of the triangle traces out the *workspace curve*.

The four-bar synthesis problem asks for the four-bar mechanisms whose workspace curve contains nine given points in the plane.

Morgan, Sommese, and Wampler numerically computed 8652 solutions to this synthesis problem; which was certified by Hauenstein, et al.

That the workspace curve contain a given point P is a single equation F_P on the nine-dimensional parameter space X. Let L be the vector space spanned by the F_P for $P \in \mathbb{R}^2$. We expect that

$$9!\,\mathrm{Vol}\,NO(L) \;=\; 8652\,.$$

Showing this would prove the root count of Morgan, Sommese, and Wampler.

Root Counting (more)

Applications of algebraic geometry abound in problems that give rise to systems of equations.

Part of the analysis of the problem is determining the number of solutions to the equations.

Traditional methods, such as intersection theory or the Bézout, Kushnirenko, or Bernstein theorems, do not always apply to these systems

The theory of Newton-Okounkov bodies provides a new tool for this task.

Computing Solutions

Perhaps more important than counting solutions are algorithms to find them.

When $\Gamma(L)$ is finitely generated, so that we have a Khovanskii basis, Anderson's flat degeneration of $\mathbf{proj}(R(L))$ into a toric variety gives a numerical algorithm to compute solutions to

$$f_1 = f_2 = \cdots = f_n = 0\,, \qquad f_i \in L\,. \qquad (*)$$

This system $(*)$ is a linear section of $\mathbf{proj}(R(L))$, and the flat degeneration transforms it into a linear section of a toric variety, whose solutions may be computed using polyhedral homotopies.

(This was sketched in 1998 by Huber-S.-Sturmfels for Grassmannians.)

There is a 'mixed volume' version for root-counting with Newton-Okounkov bodies and Anderson's degenerations also work in the 'mixed setting'. Sorting this out and implementing it is a current project with Michael Burr.