Discriminant Coamoebas in Dimension Three

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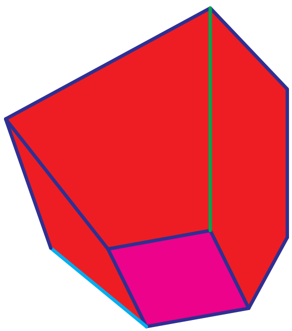
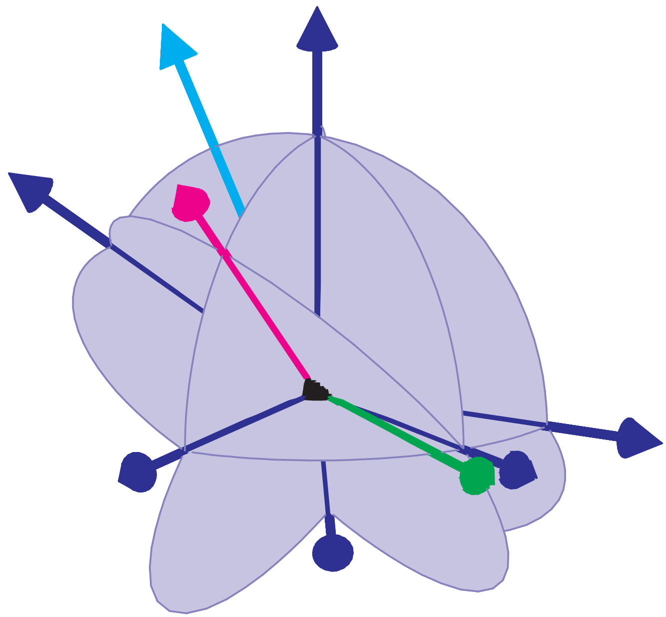




Joint work with Mounir Nisse

A-Discriminants

Let $A \subset \mathbb{Z}^n$ be a finite spanning set that lies in an affine hyperplane.

A polynomial f with support A

$$f \ := \ \sum_{a \in A} c_a x^a \qquad c_a \in \mathbb{C}\,,$$

defines a hypersurface $\mathcal{V}(f) \subset \mathbb{P}^{n-1}$.

A-discriminant : hypersurface in $\mathbb{P}^A$ of those f with $\mathcal{V}(f)$ singular.

This rational variety, its defining equation, and Newton polytope remain an object of interest. (See the talks of Forsgård.)

The A-discriminant has many homogeneities, $(\mathbb{C}^\times)^n$ acts on $\mathbb{P}^A$ and on the A-discriminant via $t.[c_a \mid a \in A] = [t^a c_a \mid a \in A]$.

Taking the quotient by $(\mathbb{C}^\times)^n$ gives the *(reduced) discriminant*.

Reduced Discriminant

Taking the quotient by $(\mathbb{C}^\times)^n$ gives the *(reduced) discriminant*, which is a hypersurface in $\mathbb{P}^{d-1}$, where $|A| = n + d$. Henceforth, this is the discriminant.

In algebraic geometry, quotients are typically badly singular and difficult to study, and the A-discriminant was already complicated.
However, Kapranov generalized the classical Horn parametrization to discriminants

$$\mathbb{C}^d \ \ni \ z \ \longmapsto \ \prod_{b\in B} \langle b, z\rangle^b \ \in \ \mathbb{P}^{d-1},$$

where B is Gale-dual to A, so that $\mathbb{Z}^d \xrightarrow{B} \mathbb{Z}^{n+d} \xrightarrow{A} \mathbb{Z}^n$ is exact.
He showed that the image is the (reduced) discriminant, D_B. Adding back the homogeneities gives the original A-discriminant.

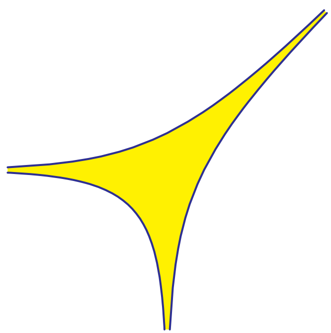
This map is central to our study of discriminants.

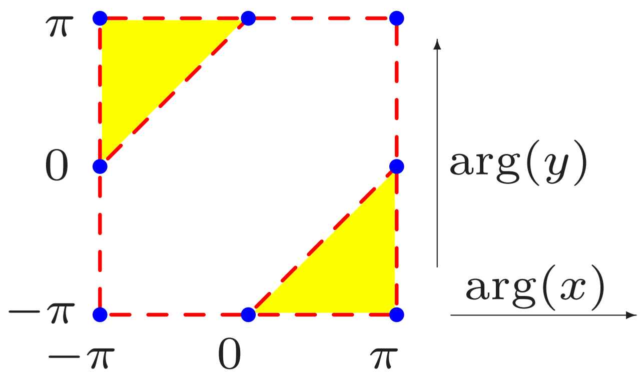
(Pre-) Tropical Objects

Let $V \subset (\mathbb{C}^\times)^n$ be a variety. Its *amoeba* $\mathcal{A}(V)$ is the set of lengths in V and its *coamoeba* $co\mathcal{A}(V)$ is the set of arguments in V. Gel'fand, Kapranov, and Zelevinsky introduced amoebas and Passare introduced coamoebas.

Formally, identify $\mathbb{C}^\times$ with $\mathbb{R} \times \mathbb{T}$, where $\mathbb{T} = S^1$ is the unit complex numbers, and $e^r\theta \mapsto (r, \theta)$. Then $\mathcal{A}(V)$ is the projection of V to $\mathbb{R}^n$ and $co\mathcal{A}(V)$ is its projection to $\mathbb{T}^n$.

Example : The amoeba and coamoeba of $\mathcal{V}(x + y + 1)$ are



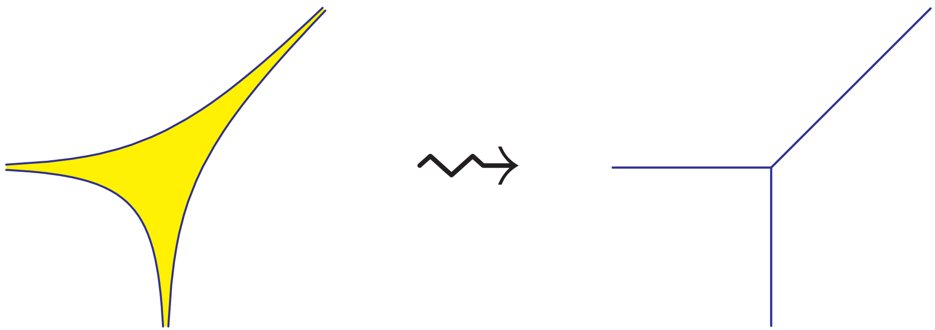


Tropical Variety

Tropical variety $\mathcal{T}(V)$ of V : cone over the limiting directions of $\mathcal{A}(V)$.

By results of Bergman and Bieri-Groves, $\mathcal{T}(V)$ is a rational polyhedral fan of the same dimension (in $\mathbb{R}^n$) as V (as a complex variety).

For our line, this is the tripod,



Geometrically, an integral weight $w \in \mathbb{Z}^n$ lies in $\mathcal{T}(V)$ if and only if the initial scheme $\lim_{t\to\infty} t^w.V \neq \emptyset$ (in $(\mathbb{C}^\times)^n$).

Tropical Objects for Discriminants

Discriminants have relatively simple and understandable tropical objects.

Theorem (Passare, Sadykov, Tsikh)
Principal A-determinants have solid amoebas. (No bounded components of complement).

We will see that the tropical variety and coamoeba of a discriminant are also surprisingly understandable.

Tropical Discriminants

Dickenstein, Feichtner, and Sturmfels found a beautiful structure theorem for tropical discriminants, using the Horn-Kapranov parametrization.

The Horn-Kapranov parametrization is a composition of two simple maps

$$\lambda_B : \mathbb{C}^d \ni z \longmapsto (\langle b, z\rangle \mid b \in B) \in \mathbb{C}^B ,$$

$$\pi_B : \mathbb{C}^B \ni x \longmapsto \prod_{b\in B} x_b^b \in \mathbb{P}^{d-1} ,$$

with λ_B linear and π_B a homomorphism on dense tori.

Their work involved two steps.

(a) Tropical variety of a linear space (Bergman fan: Sturmfels, Ardila-Klivans, and Sturmfels-Feichtner).

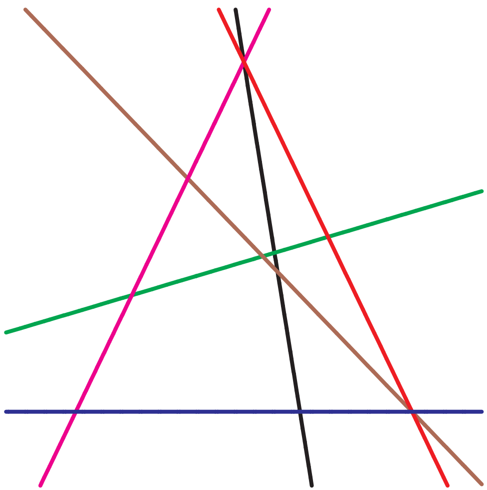
(b) Its image under the linear map induced by the homomorphism π_B.

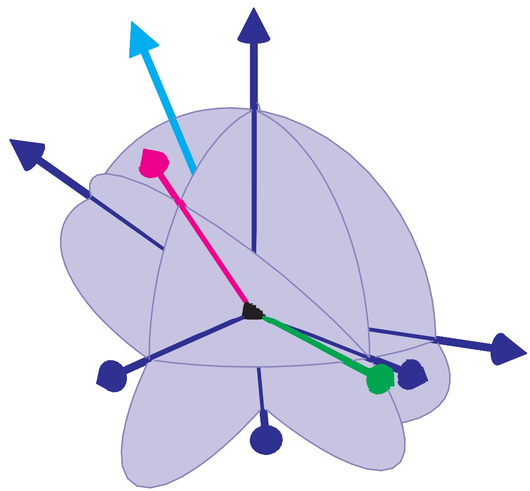
$\rightsquigarrow$ Description of tropical discriminant *and* Newton polytope of discriminant.

Example

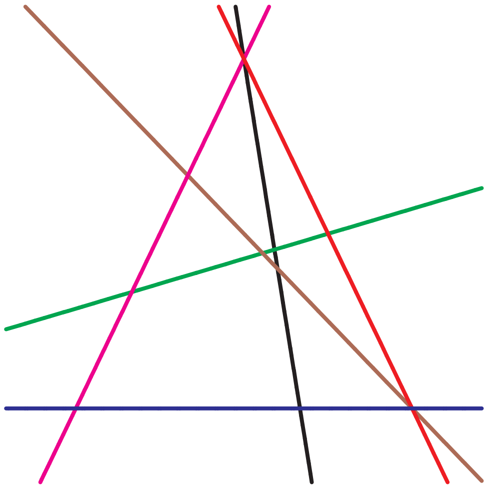
Let B be the column vectors in $\begin{pmatrix} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{pmatrix}$

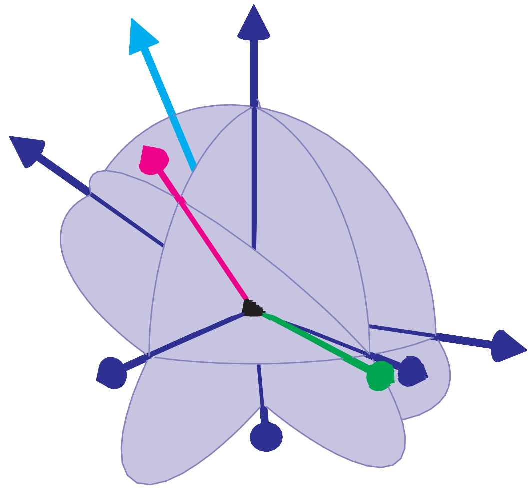
This defines a line arrangement in $\mathbb{P}^2$ and a tropical discriminant.





Example (continued)

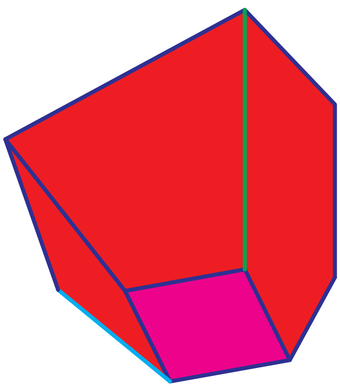


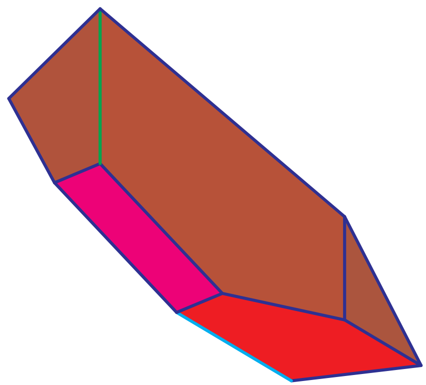


Between the line arrangement and the tropical discriminant is the Bergman fan of the arrangement. It lies in $\mathbb{R}^6/\mathbb{R}(1,\ldots,1)$ and has cones corresponding to the flags of flats of the arrangement. Its rays correspond to the irreducible flats. Here, those are the lines and points of triple intersection. The tropical discriminant is its image under the linear map given by B; it has a new ray from the intersection of two cones in the image.

Discriminant and Its Newton Polytope

$$3125q^4r^4 - 1024p^5q^2 + 1280p^4q^2r - 40p^3q^2r^2 + 4000p^2q^3r^2 - 40p^2q^2r^3 + 500pq^3r^3 + 1280pq^2r^4 + 4000q^3r^4 - 1024q^2r^5 - 432p^6 - 1152p^5q + 768p^4q^2 + 864p^5r + 1584p^4qr + 512p^3q^2r - 432p^4r^2 + 1584p^3qr^2 + 5038p^2q^2r^2 - 200pq^3r^2 - 1152p^2qr^3 + 512pq^2r^3 - 200q^3r^3 + 768q^2r^4 + 216p^5 + 832p^4q - 192p^3q^2 + 216p^4r + 532p^3qr - 208p^2q^2r + 832p^2qr^2 - 208pq^2r^2 + 16q^3r^2 - 192q^2r^3 - 27p^4 - 200p^3q + 16p^2q^2 - 200p^2qr + 16q^2r^2 + 16p^2q$$





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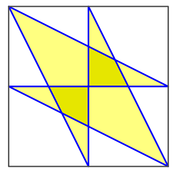
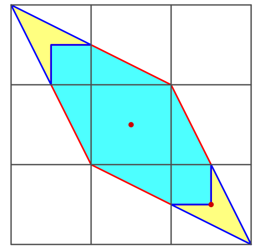
Discriminant Coamoebas in Dimension 2

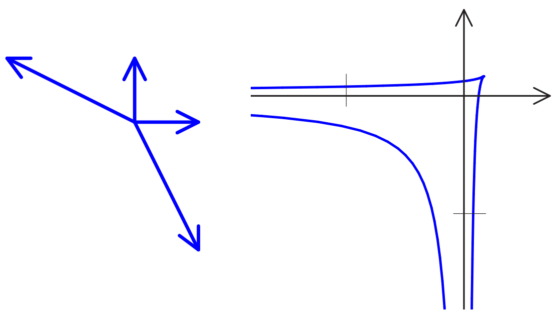
Nilsson and Passare described the *discriminant coamoeba* when $B \subset \mathbb{Z}^2$ as an explicit polyhedral object that is the complement of a zonotope generated by B in vol(A) times a fundamental domain.

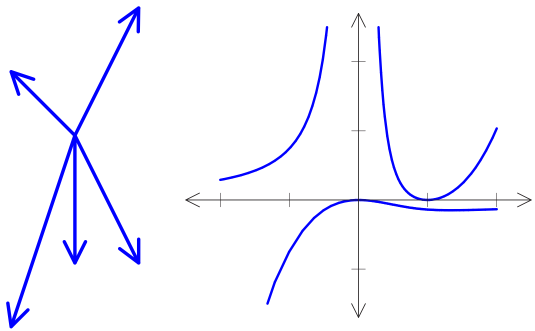
For $B = \left(\begin{smallmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{smallmatrix}\right)$, here is the coamoeba in its fundamental domain and as the complement of the zonotope.





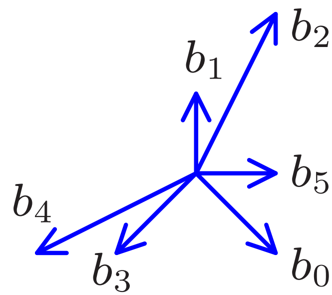
Here are some reduced dicriminants in dimension $d = 2$, together with the vectors B:





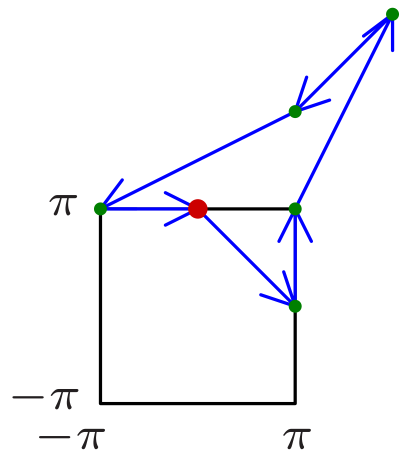
Nilsson-Passare description of discriminant coamoebae

Order the vectors of B by the clockwise order of the lines they span, starting from just below the horizontal.



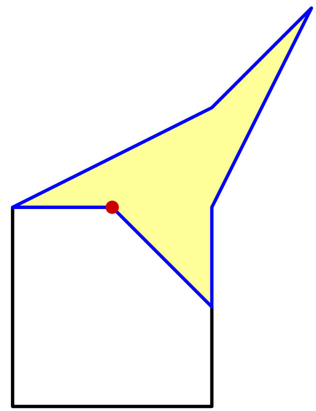
We describe the discriminant coamoeba in $\mathbb{R}^2$, the universal cover of $\mathbb{T}^2$.

Starting at which of $(0,0)$, $(\pi,0)$, $\boldsymbol{(0,\pi)}$, (π,π) is the argument of the Horn-Kapranov parametrization at $[1,t]$ for $t \gg 0$, place the vectors $\pi b_0, \ldots, \pi b_N$ in order, head-to-tail.



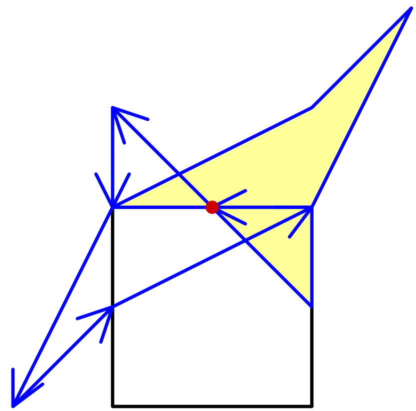
A-discriminant coamoebae

This path is the boundary of a topological 2-chain, which is the half of the coamoeba corresponding to the upper half plane.



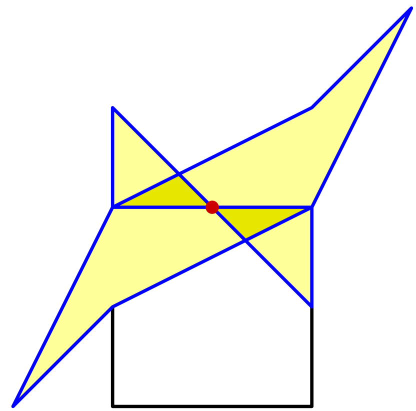
A-discriminant coamoebae

This path is the boundary of a topological 2-chain, which is the half of the coamoeba corresponding to the upper half plane. Starting again at $(0, \pi)$ place the vectors $-\pi b_0, \ldots, -\pi b_N$ in order.



A-discriminant coamoebae

This path is the boundary of a topological 2-chain, which is the half of the coamoeba corresponding to the upper half plane. Starting again at $(0, \pi)$ place the vectors $-\pi b_0, \ldots, -\pi b_N$ in order, and fill in to get the rest of the *coamoeba chain* $\mathcal{A}_B$.



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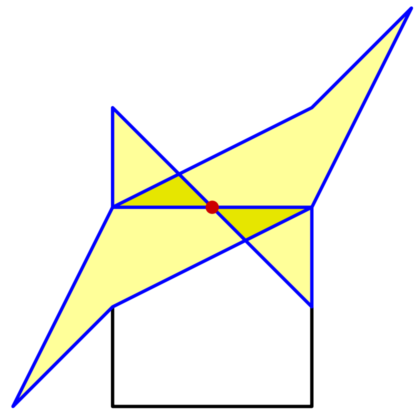

This is an oriented topological 2-chain in $\mathbb{T}^2$ whose oriented boundary consists of the edges coming from $\pi b_0, \ldots, \pi b_N, -\pi b_0, \ldots, -\pi b_n$.

A-discriminant coamoebae

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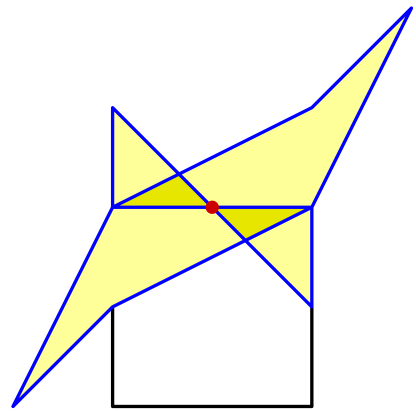
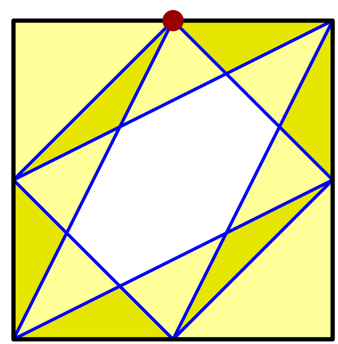
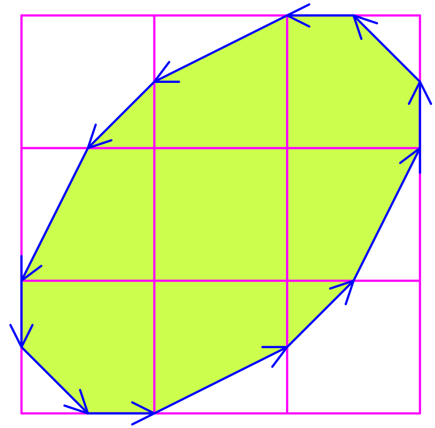


This is an oriented topological 2-chain in $\mathbb{T}^2$ whose oriented boundary consists of the edges coming from $\pi b_0, \ldots, \pi b_N, -\pi b_0, \ldots, -\pi b_n$.
We display it in the fundamental domain.

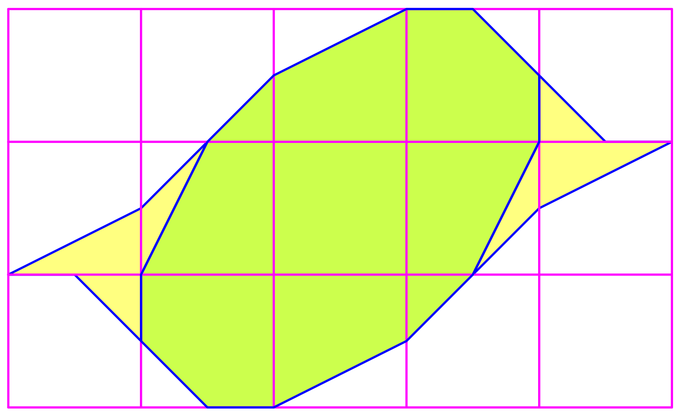


A zonotope



The boundary of this coamoeba chain, but in reverse order, is shared by the zonotope Z_B generated by the vectors $\pi b_0, \pi b_1, \ldots, \pi b_N$.

The union of Z_B and the coamoeba chain is therefore a topological 2-cycle on $\mathbb{T}^2$.



In our example, $\mathcal{A}_B \cup Z_B$ covers 7 fundamental domains.

Note that it tiles the plane.

Phase Limit Set

The *phase limit set* $\mathcal{P}^\infty(V)$ of a subvariety $V \subset (\mathbb{C}^\times)^n$ is the set of accumulation points of arguments of unbounded sequences $\{x_i \mid i \in \mathbb{N}\}$ in V. (Definition similar to the logarithmic limit set.)

Each cone σ of the tropical variety $\mathcal{T}(V)$ of V gives an initial scheme $\text{in}_\sigma V$, on which the subgroup of the torus $(\mathbb{C}^\times)^n$ corresponding to the linear span of σ acts freely. The quotient is the part of V at the infinity corresponding to σ.

Theorem (Nisse-S.)
The closure of $coA(V)$ is $co\mathcal{A}(V) \cup \mathcal{P}^\infty(V)$, and

$$\mathcal{P}^\infty(V) \;=\; \bigcup_{\sigma\neq 0} co\mathcal{A}(\text{in}_\sigma V)\,.$$

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Phase Limit Set of a Linear Space

A *flat* L of the hyperplane arrangement given by B is any intersection of hyperplanes. The set of hyperplanes $B_L \subset B$ containing L gives an arrangement on $\mathbb{C}^d/L$ and its complement B^L gives an arrangement in L. There are corresponding linear spaces $\lambda_{B_L}(\mathbb{C}^d/L) \subset \mathbb{C}^{B_L}$ and $\lambda_{B^L}(L) \subset \mathbb{C}^{B^L}$. (Deletion-restriction).

Using the definition of phase limit set, we deduce:

Theorem (Nisse-S.)
The closure of the coamoeba of $\lambda_B(\mathbb{C}^d)$ is the union of sets, one for each flat L of B. The set corresponding to L is

$$\overline{co\mathcal{A}(\lambda_{B_L}(\mathbb{C}^d/L))} \times co\mathcal{A}(\lambda_{B^L}(L)) \,.$$

The shape of this recovers what we would get from the Bergman fan using the structure theorem for phase limit sets.

More : Coamoeba of a Linear Space

A part of this is a determination of the dimension of the coamoeba (and also amoeba) of a linear space.

The dimension of the coamoeba of a linear space $\Lambda \subset \mathbb{C}^n$ of dimension k is at most the maximum of n and $2k - h(\Lambda)$, where $h(\Lambda)$ is the dimension of the maximal $(\mathbb{C}^\times)^h$ that acts on Λ, its number of homogeneities.

Lemma. $$\dim co\mathcal{A}(\Lambda) = \min\{n, 2k - h(\Lambda)\}.$$

Consequently, the top-dimensional components of the phase limit set of a linear space correspond to the *irreducible flats* of B.

The closure of the coamoeba discriminant is the image of the closure of the coamoeba of the linear space $\lambda_B(\mathbb{C}^d)$ under the homomorphism π_B.

The same is true for the phase limit set of the discriminant.

Discriminant Coamoebas in Dimension Three

Lemma If $\mathbf{dim} > \mathbf{2}$, the coamoeba discriminant equals its phase limit set.

By the structure theorem for the phase limit set, we need only understand the components coming from the rays of the tropical discriminant.

There are three types.

(a) Hyperplanes for $b \in B$. This gives a component $\mathbb{T}_b \times \overline{co\mathcal{A}(D_{B/b})}$, which we understand recursively.

(b) Other irreducible flats. If one-dimensional, the component is collapsed by π_B, and does not contribute.

(c) Rays arising from the projection π_B of the Bergman fan. These do not contribute.

In dimension three, only the first type contributes, and we obtain a description of the coamoeba as a union of polyhedra, and as lying outside of the zonotope of B, but there is more work to be done...