Irrational Toric Varieties

Special Session on Convex Bodies in
Algebraic Geometry and Representation Theory
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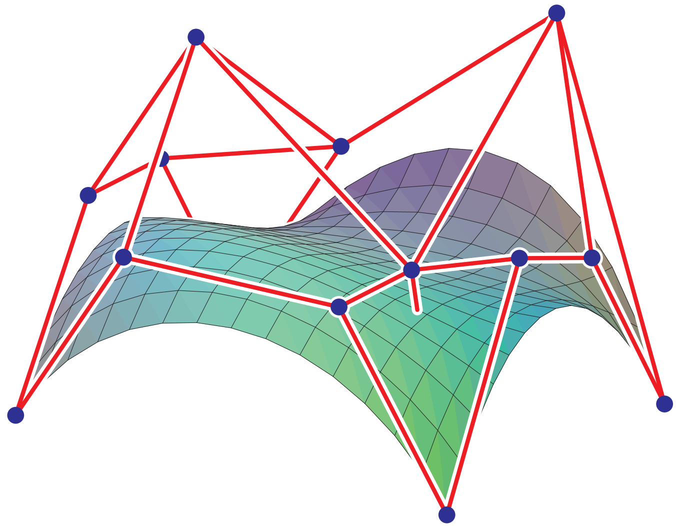

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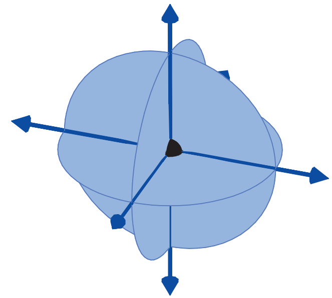
Work with: Ata Pir

Classical Toric Varieties

Let $N_{\mathbb{Z}}$ and $M_{\mathbb{Z}}$ be dual finitely generated free abelian groups.



A rational fan $\Sigma \subset N := N_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ is a polyhedral complex comprised of rational cones.

Every cone $\sigma \in \Sigma$ has a dual cone $\sigma^{\vee} \subset M$.

$S_\sigma := \sigma^{\vee} \cap M_{\mathbb{Z}}$, a finitely generated saturated submonoid of $M_{\mathbb{Z}}$.

$U_\sigma := \operatorname{spec} \mathbb{C}[S_\sigma]$ (Complex points are also $\mathrm{Hom}_{\mathrm{mon}}(S_\sigma, \mathbb{C})$.)

Toric variety associated to Σ is $X_\Sigma := \bigcup_{\sigma \in \Sigma} U_\sigma$

$(\tau \subset \sigma \Rightarrow U_\tau \subset U_\sigma)$.

Canonical positive part, $X_\Sigma(\mathbb{R}_{\geq})$, obtained by gluing $\mathrm{Hom}_{\mathrm{mon}}(S_\sigma, \mathbb{R}_{\geq})$.

Pre-Classical Toric Varieties

An element $a \in M \rightsquigarrow$ homomorphism $N \ni u \mapsto \exp(\langle a, u\rangle) \in \mathbb{R}_>$.

Let $\mathcal{A} \subset M$ be a finite set that we assume spans (for simplicity).
This gives a map $\varphi_\mathcal{A} \colon N \to (\mathbb{R}_\geq)^\mathcal{A}$ by $u \mapsto (\exp(\langle a, u\rangle) \mid a \in \mathcal{A})$.

Define $Y_\mathcal{A} := \overline{\varphi_\mathcal{A}(N)}$ (usual closure), an *irrational affine toric variety*.

Note that $\mathrm{cone}(\mathcal{A})$ is tautologically parametrized by the orthant $(\mathbb{R}_\geq)^\mathcal{A}$;
$\pi_\mathcal{A} \colon (\mathbb{R}_\geq)^\mathcal{A} \to \mathrm{cone}(\mathcal{A})$, where $(x_a \mid a \in \mathcal{A}) \mapsto \sum_a x_a \cdot a$.

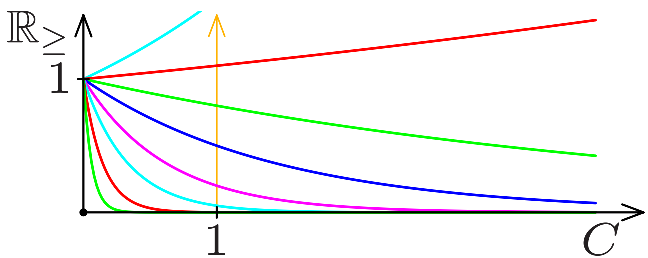
Theorem. (Birch 1963)
The tautological map $\pi_\mathcal{A} \colon Y_\mathcal{A} \to \mathrm{cone}(\mathcal{A})$ is a homeomorphism of singular manifolds with boundary.

Spaces of Monoid Maps

A polyhedral cone $C \subset M$ is an additive monoid.

Definition. $\mathrm{Hom}_c(C, \mathbb{R}_\geq)$ is the set of monoid homomorphisms, maps $\varphi\colon C \to \mathbb{R}_\geq$ with $\varphi(0) = 1$ such that $\varphi(a+b) = \varphi(a)\varphi(b)$, that are continuous on the relative interior of every face.
Equip this with the weak topology, where point evaluations are continuous.

Example. Let $C := [0, \infty) \subset \mathbb{R}$.
Elements $\varphi \in \mathrm{Hom}_c(C, \mathbb{R}_\geq)$ have the form $\varphi(r) = \varphi(1)^r$.



The map $\varphi \mapsto \varphi(1)$ identifies $\mathrm{Hom}_c(C, \mathbb{R}_\geq)$ with C.

Affine Toric Varieties From Cones

We have that $N \simeq \mathrm{Hom}_c(M, \mathbb{R}_>)$, as topological groups.
Write T_N for $\mathrm{Hom}_c(M, \mathbb{R}_>) = \mathrm{Hom}_c(M, \mathbb{R}_\geq)$.

The diagonal map $u \mapsto (u, u)$ is a monoid map $C \to C \oplus M$, which induces an action $T_N \times \mathrm{Hom}_c(C, \mathbb{R}_\geq) \to \mathrm{Hom}_c(C, \mathbb{R}_\geq)$.

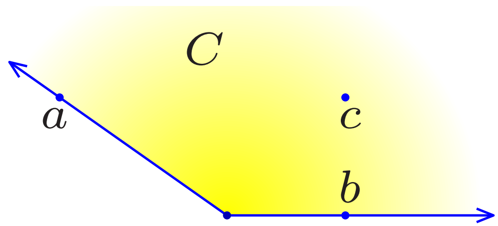
The example when $C = [0, \infty)$ generalizes.

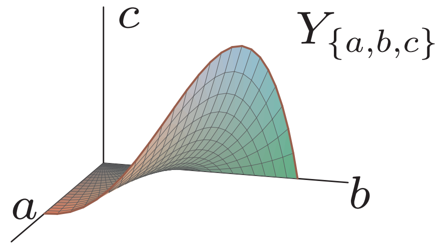
Theorem. *Let $\mathcal{A}$ be a generating set for the cone C.*
Then $\varphi \mapsto (\varphi(a) \mid a \in \mathcal{A}) \in (\mathbb{R}_\geq)^{\mathcal{A}}$ induces an equivariant homeomorphism between $\mathrm{Hom}_c(C, \mathbb{R}_\geq)$ and $Y_{\mathcal{A}}$.
Composing with the tautological map shows $\mathrm{Hom}_c(C, \mathbb{R}_\geq) \simeq C$.

The T_N-orbits are the relative interiors of the faces.

Example

Example. Suppose that $C := \text{cone}\{a, b, c\}$ where $a := (-\sqrt{2}, 1)$, $b := (1, 0)$, and $c := (1, 1)$.





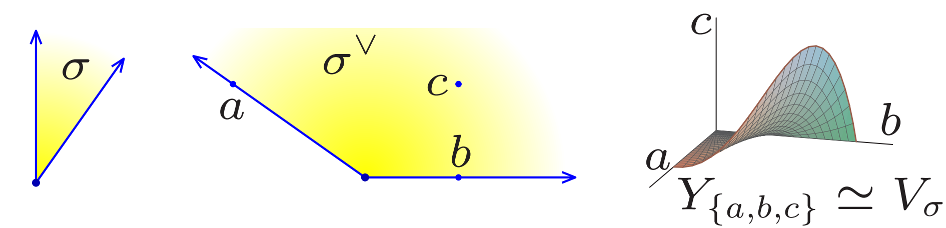
Irrational Toric Varieties

Let $\Sigma \subset N$ be any fan (not necessarily rational).

A cone $\sigma \in \Sigma$ has a dual cone $\sigma^\vee \subset M$.

Define $V_\sigma := \mathrm{Hom}_c(\sigma^\vee, \mathbb{R}_\geq)$. As before, $\tau \subset \sigma \Rightarrow V_\tau \subset V_\sigma$.

Example. Let $\sigma := \mathrm{cone}\{(1, \sqrt{2}), (0, 1)\}$. Then $\sigma^\vee$ is the cone of the previous example.



Definition. The irrational toric variety Y_Σ is the union $\bigcup_{\sigma \in \Sigma} V_\sigma$.

Properties of Irrational Toric Varieties

Theorem. *Let $\Sigma \subset N$ be a fan.*

- *Y_Σ is a T_N-equivariant cell complex. Each cell is a T_N-orbit isomorphic to $N/\langle\sigma\rangle$ for a unique cone σ of Σ. The inclusion of cells is dual to that of Σ.*
- *The topological space Y_Σ is complete if and only if the fan Σ is complete.*
- *When Σ is a rational fan, then $Y_\Sigma = X_\Sigma(\mathbb{R}_\geq)$, the positive part of the classical toric variety X_Σ.*
- *Y_Σ has an equivariant embedding in a simplex $\triangle^n$ (positive part of $\mathbb{P}^n$) if and only if Σ is the normal fan to a polytope Δ, and in that case $Y_\Sigma \simeq \Delta$.*
- *The association $\Sigma \mapsto Y_\Sigma$ is functorial for maps of fans.*

Resolution: GIT for Simplices?

When $\mathcal{A} \subset M$ lies on an affine hyperplane, then $Y_{\mathcal{A}}$ is a cone, and we may consider $Y_{\mathcal{A}} \subset \triangle^{\mathcal{A}}$ to be an irrational projective toric variety.

The torus $T_{\mathcal{A}}$ for $\triangle^{\mathcal{A}}$ acts on $Y_{\mathcal{A}}$.

A question in geometric modeling led us[1] to try to describe all limits in the Hausdorff sense of sequences of translates of $Y_{\mathcal{A}}$ by elements of $T_{\mathcal{A}}$. When $\mathcal{A} \subset M_{\mathbb{Z}}$, they are all toric degenerations and the Hausdorff space may be identified with the secondary polytope of $\mathcal{A}$.

For general $\mathcal{A}$, new methods were developed[2] to identify all Hausdorff limits as toric degenerations, but we could not identify the Hausdorff space.

This theory was developed to enable the identification, showing that the Hausdorff space is the irrational toric variety of the secondary fan of $\mathcal{A}$.

[1] w/ García-Puente, Craciun, Zhu
[2] w/ Postinghel and Villamizar