Numerical Irreducible Decomposition for Multiprojective Varieties

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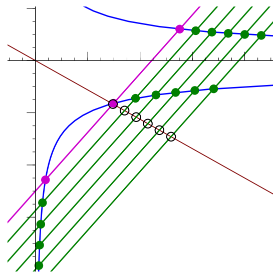
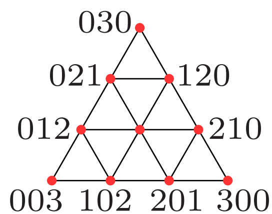
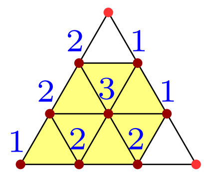
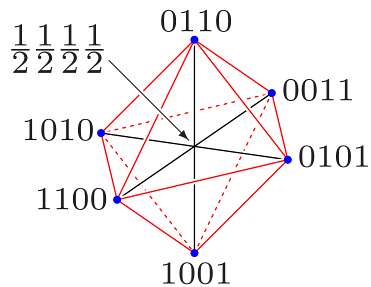








Work with Jon Hauenstein, Anton Leykin, and Jose Rodriguez.

Numerical Irreducible Decomposition

In numerical algebraic geometry, a (reduced) variety $V \subset \mathbb{P}^n$ of dimension m is represented by a witness set $W := V \cap M$, for $M \subset \mathbb{P}^n$ a general linear subspace of codimension m. Note: $\deg(V) = \#W$.

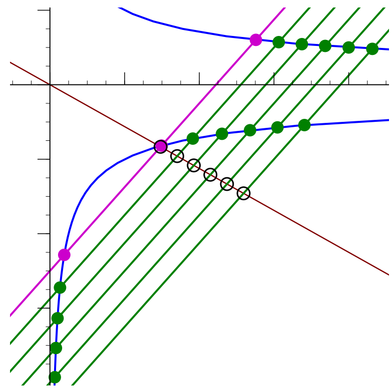
Numerical irreducible decomposition computes the partition $W = W^1 \sqcup W^2 \sqcup \cdots \sqcup W^s$, where $W^i := V^i \cap M$, for $V^1, \ldots, V^s$ the irreducible components of V.

Two steps:

(1) Monodromy. Move the slice M in loops to compute a partition

$$W = U^1 \sqcup U^2 \sqcup \cdots \sqcup U^t,$$

where each U^i lies in one component V^{a_i} of V.



(2) Use trace test to verify $U^i = V^{a_i} \cap M$.

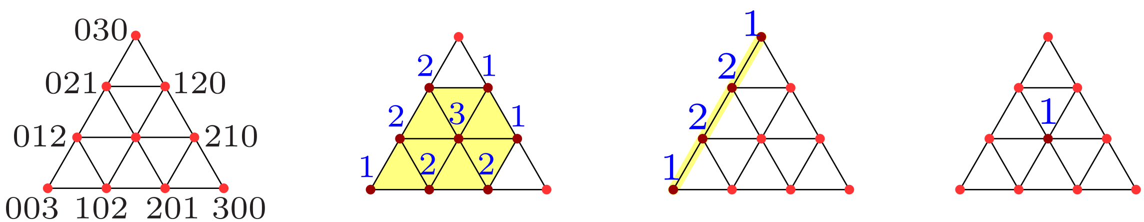
Multiprojective varieties

$V \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ is a *multiprojective variety*. It is defined by polynomials that are separately homogeneous in r sets of variables.

For $\mathbf{m} = (m_1, \ldots, m_r)$ with $0 \leq m_i \leq n_i$ and $m_1 + \cdots + m_r = m = \dim(V)$, let $M^{(\mathbf{m})} := M_1 \times \cdots \times M_r$ where $M_i \subset \mathbb{P}^{n_i}$ is a general linear subspace of codimension m_i.

The *multidimension* $\mathrm{Dim}(V)$ of V are those $\mathbf{m}$ such that $V \cap M^{(\mathbf{m})} \neq \emptyset$, and its $\mathbf{m}$*-th multidegree* is $\mathrm{Deg}_V(\mathbf{m}) := \#V \cap M^{(\mathbf{m})}$.

Here are some multidimensions of 3-folds in of $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$.



Witness Collections

Hauenstein and Rodriguez: a *witness collection* for V is the multilinear sections $\{V \cap M^{(\mathbf{m})} \mid \mathbf{m} \in \text{Dim}(V)\}$.

Reflects the structure of V and exponentially smaller than the witness set for the Segre embedding of V into $\mathbb{P}^{\prod(n_i+1)-1}$.

Many algorithms operate on a witness collection, except numerical irreducible decomposition. The trace is not multilinear, and the trace test requires an affine or projective embedding.

Because of complexity, the Segre embedding is unsuitable, and it is not clear how to pass to an affine patch $\mathbb{C}^{n_1+\cdots+n_r}$.

However, when V is a curve or $r = 2$ neither of these is an issue.

I will sketch a numerical toolbox for reducing numerical irreducible decomposition for multiprojective varieties to that for curves.

Equidimensional Decomposition

When V is irreducible and $x \in V$ is general, $\text{Dim}(V) = \text{Dim}(T_x V)$.

This gives a multiprojective local dimension test, as $\text{Dim}(T_x V)$ is computed from the ranks of submatrices of the Jacobian matrix for equations for V.

Given a witness collection W for a possibly reducible multiprojective variety V, the local dimension test partitions the points of W by the multidimension of the component of V on which they lie.

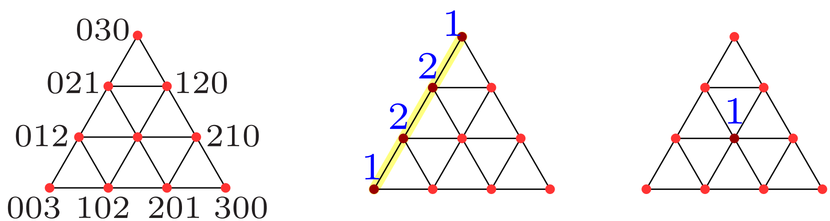
Thus we may assume that W is a witness collection for an *equidimensional* multiprojective variety.

The possible sets $\text{Dim}(V)$ for V irreducible were determined by Castillo, *et al.* to be the integer points in a polymatroid polytope determined by V.

Products

When V is equidimensional and $\mathrm{Dim}(V) = P \times Q$ is a product, each irreducible component of V is a product $Y \times Z$, where Y and Z lie in disjoint factors of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, with $\mathrm{Dim}(Y) = P$ and $\mathrm{Dim}(Z) = Q$, and witness sets for each multidimension $\mathbf{m}$ are also products.

Consider the two components with multidimension and degree,



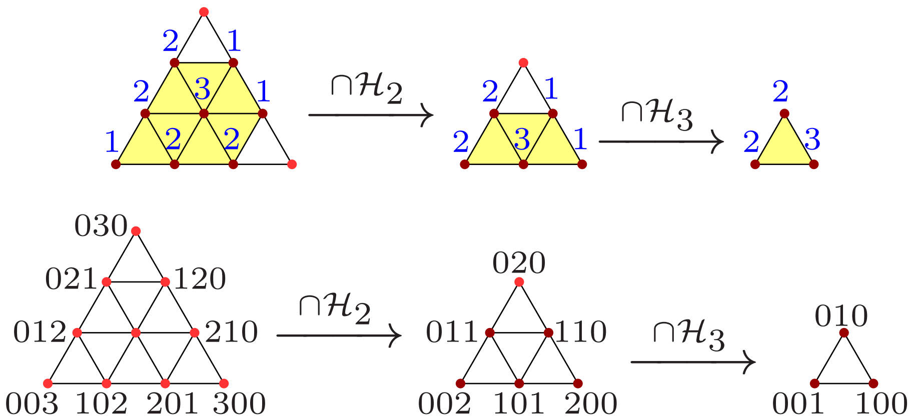
The one on the left is the product of a point $p \in \mathbb{P}^3$ (in the first factor) and a 3-fold $Y \subset \mathbb{P}^3 \times \mathbb{P}^3$, and the one on the right is the product $\ell_1 \times \ell_2 \times \ell_3$, where ℓ_i is a line in the ith factor of $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$.

Slicing

Assume V is equidimensional and is not a product.

If $\pi_i : V \to \mathbb{P}^{n_i}$ has $\dim \pi_i(V) > 1$, then Bertini's Theorem implies that $V \cap \mathcal{H}_i$ has the same irreducible decomposition as V. ($\mathcal{H}_i$ is a hyperplane section from $\mathbb{P}^{n_i}$.)

We use this to reduce to the case where each $\pi_i(V)$ is a curve, which is simply bookkeeping.



Merging

If V is not a curve, some projection to a biproduct $\mathbb{P}^{n_i} \times \mathbb{P}^{n_j}$ is a surface.

Passing to an affine patch $\mathbb{C}^{n_i+n_j}$, then to $\mathbb{P}^{n_i+n_j}$, and then slicing in this factor preserves irreducibility, reduces dimension, and merges witness sets.

Let $V \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be given by a general multi-linear form and by $5+4x_4^4+3x_3^3+2x_2^2+x_1$.