HOMEWORK 1 MATH 689 SECTION 604

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Hand in two problems in each section. Due February 28. Let $\mathbb{F}$ be an arbitrary field.

Finger Exercises

(1) Verify the claim that smallest ideal containing a set $S \subset \mathbb{F}[x_1, \ldots, x_n]$ of polynomials is the set of all expressions of the form $h_1 f_1 + \cdots + h_m f_m$ where $f_1, \ldots, f_m \in S$ and $h_1, \ldots, h_m \in \mathbb{F}[x_1, \ldots, x_n]$.

(2) Let $\mathcal{I}$ be an ideal of $\mathbb{C}[x_1, \ldots, x_n]$. Show that

$$\sqrt{\mathcal{I}} \ := \ \{f \in \mathbb{F}[x_1, \ldots, x_n] \mid f^m \in \mathcal{I} \text{ for some } m \geq 1\}$$

is an ideal, is radical, and is the smallest radical ideal containing $\mathcal{I}$.

(3) Prove that in $\mathbb{A}^2$, we have $\mathcal{V}(y - x^2) = \mathcal{V}(y^3 - y^2x^2, x^2y - x^4)$.

(4) Express the cubic space curve C with parametrization (t, t^2, t^3) in each of the following ways.
   (a) The intersection of a quadric and a cubic hypersurface.
   (b) The intersection of two quadrics.
   (c) The intersection of three quadrics.

Zariski topology

(5) Show that no proper nonempty open subset S of $\mathbb{R}^n$ or $\mathbb{C}^n$ is a variety. Here, we mean open in the usual (Euclidean) topology on $\mathbb{R}^n$ and $\mathbb{C}^n$. (Hint: Consider the Taylor expansion of any polynomial in $\mathcal{I}(S)$.)

(6) (a) Describe all the algebraic varieties in $\mathbb{A}^1_{\mathbb{F}}$.
   (b) Show that any open set in $\mathbb{A}^1_F \times \mathbb{A}^1_F$ is open in $\mathbb{A}^2_{\mathbb{F}}$.
   (c) Find a Zariski open set in $\mathbb{A}^2_{\mathbb{F}}$ which is not open in $\mathbb{A}^1_{\mathbb{F}} \times \mathbb{A}^1_{\mathbb{F}}$.

(7) (a) Show that the Zariski topology in $\mathbb{A}^n_{\mathbb{F}}$ is not Hausdorff if $\mathbb{F}$ is infinite.
   (b) Prove that any nonempty open subset of $\mathbb{A}^n_{\mathbb{F}}$ is dense.
   (c) Prove that $\mathbb{A}^n_{\mathbb{F}}$ is compact.

(8) Show that a regular map $\varphi\colon X \to Y$ between affine varieties X and Y is continuous in the Zariski topology.

Algebraic varieties.

(9) Let $\mathbb{A}^{n^2}_{\mathbb{F}}$ be the set of $n \times n$ matrices.
  (a) Show that the set $\mathbf{SL}(n, \mathbb{F}) \subset \mathbb{A}^{n^2}_{\mathbb{F}}$ of matrices with determinant 1 is an algebraic variety.
  (b) Show that the set of singular matrices in $\mathbb{A}^{n^2}_{\mathbb{F}}$ is an algebraic variety.
  (c) Show that the set $\mathbf{GL}(n, \mathbb{C})$ of invertible matrices is not an algebraic variety in $\mathbb{C}^{n^2}$. $GL_n(\mathbb{F})$ can be identified with an algebraic subset of $\mathbb{C}^{n^2+1} = \mathbb{C}^{n^2} \times \mathbb{C}^1$. Find the corresponding map.

(10) An $n \times n$ matrix with complex entries is *unitary* if its columns are orthonormal under the complex inner product $\langle z, w \rangle = z \cdot \overline{w}^t = \sum_{i=1}^n z_i \overline{w_i}$. Show that the set $\mathbf{U}(n)$ of unitary matrices is not a complex algebraic variety. Show that it can be described as the zero locus of a collection of polynomials with real coefficients in $\mathbb{R}^{2n^2}$, and so it is a real algebraic variety.

(11) Let $\mathbb{A}^{mn}_{\mathbb{F}}$ be the set of $m \times n$ matrices over $\mathbb{F}$.
  (a) Show that the set of matrices of rank $\leq r$ is an algebraic variety.
  (b) Show that the set of matrices of rank $= r$ is not an algebraic variety if $r > 0$.

(12) (a) Show that the set $\{(t, t^2, t^3) \mid t \in \mathbb{F}\}$ is an algebraic variety in $\mathbb{A}^3_{\mathbb{F}}$.
  (b) Show that the following sets are not algebraic varieties
    (i) $\{(x, y) \in \mathbb{A}^2_{\mathbb{R}} | y = \sin x\}$
    (ii) $\{(\cos t, \sin t, t) \in \mathbb{A}^3_{\mathbb{R}} \mid t \in \mathbb{R}\}$
    (iii) $\{(x, e^x) \in \mathbb{A}^2_{\mathbb{R}} \mid x \in \mathbb{R}\}$

Algebra - Geometry dictionary

(13) Let I be an ideal in $R = \mathbb{F}[x_1, \dots, x_n]$. Prove or find counterexamples to the following statements.
  (a) If $V(I) = \mathbb{A}^n_{\mathbb{F}}$ then $I = (0)$.
  (b) If $V(I) = \emptyset$ then $I = R$.

(14) (a) Give an example of two algebraic varieties V and W such that $I(V \cap W) \neq I(V) + I(W)$.
  (b) Show that $f(x, y) = y^2 + x^2(x-1)^2 \in \mathbb{R}[x, y]$ is an irreducible polynomial but that $V(f)$ is reducible.

(15) Let $f, g \in \mathbb{F}[x, y]$ be coprime polynomials. Show that $V(f) \cap V(g)$ is a finite set.

(16) (a) Let $R = \mathbb{F}[x_1, x_2, \dots, x_n]$ and I be an ideal of R. Show that if R/I is a finite dimensional $\mathbb{F}$-vector space then $V(I)$ is a finite set.
  (b) Let $J = (xy, yz, xz)$ be an ideal in $\mathbb{F}[x, y, z]$. Find the generators of $I(V(J))$. Show that J cannot be generated by two polynomials in $\mathbb{F}[x, y, z]$. Describe $V(I)$ where $I = (xy, xz - yz)$. Show that $\sqrt{I} = J$.

(17) Prove that there are three points p, q, r in $\mathbb{A}^2_{\mathbb{F}}$ such that

$$\sqrt{x^2 - 2xy^4 + y^6, y^3 - y} = I(\{p\}) \cap I(\{q\}) \cap I(\{r\}).$$

Find a reason why you would know that the ideal $(x^2 - 2xy^4 + y^6, y^3 - y)$ is not a radical ideal.

Projective algebraic varieties

(18) (a) Show that two distinct lines in $\mathbb{P}^2$ always intersect in one point.
(b) Let n be a positive integer and $\mathbb{P}^n$ be projective n-space over $\mathbb{F}$. Show that there is a natural decomposition $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \cdots \cup A^1 \cup A^0$ into disjoints subsets. Compute the number of elements of $\mathbb{P}^n$ when $\mathbb{F}$ is a finite field of q elements.

(19) For any $d \in \mathbb{Z}_{\geq 0}$, let $S_d \subset \mathbb{F}[x_0, x_1, \ldots, x_n]$ be the $\mathbb{F}$-vector space of homogeneous polynomials of degree d. Prove that $\dim_{\mathbb{F}} S_d = \binom{d+n}{n}$.

(20) A *conic* is a variety defined by a quadratic equation.
(a) Show that in $\mathbb{R}^2$ there are eight types of conics. Give an example of each type.
(b) Show that in $\mathbb{A}^2_{\mathbb{C}}$ there are only five types of conics.
(c) Show that in $\mathbb{P}^2_{\mathbb{C}}$ there are only three types of conics. Hint: This is a classification by the *rank* of a conic, where the rank of a quadratic form $\sum_i a_{ii}x_i^2 + 2\sum_{i<j} a_{ij}x_ix_j$ is defined by the rank of the symmetric matrix (a_{ij}).

Maps of affine and projective varieties

(21) Let $C = V(y^2 - x^3)$ show that the map $\phi : \mathbb{A}^1_{\mathbb{C}} \to C$, $\phi(t) = (t^2, t^3)$ is a homeomorphism in the Zariski topology but it is not an isomorphism of affine varieties.

(22) Let $V = V(y - x^2) \subset \mathbb{A}^2_{\mathbb{F}}$ and $W = V(xy - 1) \subset \mathbb{A}^2_{\mathbb{F}}$. Show that

$$\mathbb{F}[V] := \mathbb{F}[x, y]/I(V) \cong \mathbb{F}[t]$$

$$\mathbb{F}[W] := \mathbb{F}[x, y]/I(W) \cong \mathbb{F}[t, \frac{1}{t}]$$

Conclude that the hyperbola $V(xy - 1)$ is not isomorphic to the affine line.

(23) Let $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ be given by $\phi([x_0 : x_1]) = [x_0^2 : x_0x_1 : x_1^2]$. Show that $C = \phi(\mathbb{P}^1)$ and $\mathbb{P}^1$ are isomorphic as projective varieties but their coordinate rings are not.

(24) Give an isomorphism between $\mathbb{P}^1$ and $V(x^2 + y^2 - z^2) \subset \mathbb{P}^2$. Use this to parametrize all integer solutions to the equation $x^2 + y^2 = z^2$.

Dimension

(25) Fix the hyperbola $H = V(xy - 5) \subset \mathbb{A}^2_{\mathbb{R}}$ and let C_t be the circle $x^2 + (y - t)^2 = 1$ for $t \in \mathbb{R}$.
(a) Show that $H \cap C_t$ is zero-dimensional, for any choice of t.
(b) Determine the number of points in $H \cap C_t$ (this number depends on t).

(26) Let V be a d-dimensional irreducible affine variety in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$ such that $V \cap H \neq \emptyset$ and $V \cap H \neq V$. Show that all irreducible components of $V \cap H$ have dimension $d - 1$.

(27) Let I be an ideal of $\mathbb{F}[x_1, \ldots, x_n]$ which can be generated by r elements. Prove that every irreducible component of $V(I)$ has dimension $\geq n - r$.

(28) Show that an irreducible affine variety is zero-dimensional if and only if it is a point.

(29) Show that $\mathbb{A}^2$ has dimension 2, using the combinatorial definition of dimension.