Existence of Large Singular Solutions of Conformal Scalar Curvature Equations in S^n

Steven D. Taliaferro
Mathematics Department
Texas A&M University
College Station, TX 77843-3368
stalia@math.tamu.edu

Abstract

We prove that every positive function in $C^1(S^n)$, $n \geq 6$, can be approximated in the $C^1(S^n)$ norm by a positive function $K \in C^1(S^n)$ such that the conformal scalar curvature equation

$$-\Delta u + \frac{n(n-2)}{4} u = K u^{\frac{n+2}{n-2}} \qquad \text{in} \quad S^n \tag{0.1}$$

has a weak positive solution u whose singular set consists of a single point. Moreover, we prove there does not exist an apriori bound on the rate at which such a solution u blows up at its singular point.

Our result is in contrast to a result of Caffarelli, Gidas, and Spruck which states that equation (0.1), with K identically a positive constant in S^n, $n \geq 3$, does not have a weak positive solution u whose singular set consists of a single point.

1 Introduction and statement of results

In this paper we study the existence of positive functions $K \in C^1(S^n)$ such that the conformal scalar curvature equation

$$-\Delta u + \frac{n(n-2)}{4} u = K u^{n^*} \quad \text{in} \quad S^n,\ n \geq 3, \tag{1.1}$$

has a weak positive solution u whose singular set consists of a single point, where $n^* = (n+2)/(n-2)$. Moreover, given any large continuous function $\varphi\colon (0,1) \to (0,\infty)$, we investigate when such a solution u can be required to satisfy

$$u(P) \neq O(\varphi(|P-Q|)) \quad \text{as} \quad P \to Q, \tag{1.2}$$

where $\{Q\}$ is the singular set of u.

By a weak positive solution u of (1.1), we mean a positive function $u \in L^{n^*}(S^n)$ such that

$$-\int_{S^n} u\Delta\zeta + \frac{n(n-2)}{4}\int_{S^n} u\zeta = \int_{S^n} K u^{n^*}\zeta \quad \text{for all} \quad \zeta \in C^\infty(S^n).$$

By the singular set of a weak positive solution u of (1.1), we mean the set of all points Q in S^n such that u is unbounded in every neighborhood of Q. If Q does not belong to the singular set of a weak positive soluton u of (1.1), then, by standard elliptic theory, u is C^2 in a neighborhood of Q.

Our main result is the following theorem.

Theorem 1. *Let $\varphi\colon (0,1) \to (0,\infty)$ be a continuous function. Then every positive function $\kappa \in C^1(S^n)$, $n \geq 6$, can be approximated in the $C^1(S^n)$ norm by a positive function $K \in C^1(S^n)$ such that for some $Q \in S^n$ there exists a weak positive solution u of (1.1) and (1.2) whose singular set is $\{Q\}$. Furthermore, given a positive number ε, the function K can also be required to satisfy $K(P) = \kappa(P)$ for $|P - Q| \geq \varepsilon$.*

Theorem 1 is in contrast to a result of Caffarelli, Gidas, and Spruck [1] which states that equation (1.1) with K identically a positive constant in S^n, $n \geq 3$, does not have a weak positive solution whose singular set consists of a single point. Moreover, the conclusion of Theorem 1 that the function u can be required to satisfy (1.2) is in contrast to another result of theirs which states that a C^2 positive solution of

$$-\Delta u + \frac{n(n-2)}{4} u = u^{n^*}$$

in a punctured neighborhood of some point Q in S^n must satisfy

$$u(P) = O\left(|P - Q|^{\frac{-(n-2)}{2}}\right) \quad \text{as} \quad P \to Q. \tag{1.3}$$

When K is identically a positive constant in S^n, Schoen [8] proved the existence of a weak positive solution of (1.1) whose singular set is any prescribed finite subset of S^n consisting of at least *two* points, and Chen and Lin [2] proved, when $n \geq 9$, the existence of a weak positive solution of (1.1) whose singular set is S^n. Later, Mazzeo and Pacard [7] gave another proof of Schoen's result.

Theorem 1 is not true when n is 3 or 4 because if u is a C^2 positive solution of

$$-\Delta u + \frac{n(n-2)}{4} u = K u^{n^*} \tag{1.4}$$

in some punctured neighborhood of some point $Q \in S^n$, then Chen and Lin [3] proved that u satisfies (1.3) when $n = 3$ and K is positive and Hölder continuous with exponent $\alpha > 1/2$ in some neighborhood of Q, and Taliaferro and Zhang [10] proved that u satisfies (1.3) when $n = 4$ and K is positive and C^1 in some neighborhood of Q. An open question is whether Theorem 1 is true when $n = 5$.

When $\kappa \equiv 1$ in S^n, $n \geq 3$, the analog of Theorem 1 concerning the approximation of κ in the $C^0(S^n)$ norm instead of the $C^1(S^n)$ norm is true. In fact, Taliaferro and Zhang [9] proved the following much stronger result.

Theorem A. *Let $Q \in S^n$, $n \geq 3$, and let $\varphi\colon (0,1) \to (0,\infty)$ and $k\colon S^n \to (0,1]$ be continuous functions such that $k(Q) = 1$ and k is less than 1 on a sequence of points in $S^n - \{Q\}$ which tends to Q. Then there exists $K \in C^0(S^n)$ satisfying $k \leq K \leq 1$ such that (1.4) has a C^2 positive solution in $S^n - \{Q\}$ satisfying (1.2).*

Leung [5] proved a result very similar to Theorem A and he also proved the existence of a positive *Lipschitz* continuous function K on S^n, $n \geq 5$, such that (1.4) has a C^2 positive solution in $S^n - \{Q\}$ not satisfying (1.3).

Lin [6] proved that if u is a C^2 positive solution of (1.4) in some punctured neighborhood of some point $Q \in S^n$, where K is a C^1 positive function in some neighborhood of Q satisfying $\nabla K(Q) \neq 0$, then u satisfies (1.3). Thus the point Q in Theorem 1 must be a critical point of K when $r^{\frac{n-2}{2}} \varphi(r) \to \infty$ as $r \to 0^+$.

Since the function u in Theorem 1 satisfies (1.2) where no bound is imposed on the size of φ near 0, one might think that the largest subset of S^n in which u could be a weak positive solution of (1.4) would be $S^n - \{Q\}$ and therefore the conclusion of Theorem 1 that u is a weak positive solution in S^n would be impossible. However this is not the case. Indeed, if u is any C^2 positive solution of (1.4) in some punctured neighborhood $\mathcal{O}$ of some point $Q \in S^n$ then $u \in L^{n^*}_{\text{loc}}(\mathcal{O} \cup \{Q\})$ and u is a weak solution of (1.4) in $\mathcal{O} \cup \{Q\}$. (See [1, Lemma 2.1] or [4, Lemma 1].)

To prove Theorem 1, choose $Q \in S^n$ such that $\nabla\kappa(Q) = 0$ and let π be the stereographic projection of S^n onto $\mathbf{R}^n \cup \{\infty\}$ which takes Q to the origin in $\mathbf{R}^n$. It is well-known that u is a weak positive solution of (1.1) with singular set $\{Q\}$ if and only if

$$v(x) := \left(\frac{2}{|x|^2+1}\right)^{\frac{n-2}{2}} u(\pi^{-1}(x)), \quad x \in \mathbf{R}^n - \{0\},$$

is a C^2 positive solution of

$$\begin{aligned} -\Delta v &= K(x)v^{n^*} \quad \text{in} \quad \mathbf{R}^n - \{0\} \\ v(x) &= O(|x|^{2-n}) \quad \text{as} \quad |x| \to \infty \\ v(x) &\neq O(1) \quad \text{as} \quad |x| \to 0^+. \end{aligned}$$

Therefore, in order to prove Theorem 1, it suffices to prove the following theorem concerning the equation

$$-\Delta u = K(x)u^{n^*} \quad \text{in} \quad \mathbf{R}^n - \{0\}, \quad n \geq 6, \tag{1.5}$$

where $n^* = (n+2)/(n-2)$.

Theorem 2. *Suppose* $\kappa \colon \mathbf{R}^n \to \mathbf{R}$ *is a* C^1 *function which is bounded between positive constants and satisfies* $\nabla\kappa(0) = 0$. *Let* ε *be a positive number and let* $\varphi \colon (0,1) \to (0,\infty)$ *be a continuous function. Then there exists a* C^1 *function* $K \colon \mathbf{R}^n \to \mathbf{R}$ *satisfying* $\nabla K(0) = 0$, $K(x) = \kappa(x)$ *for* $|x| \geq \varepsilon$, $K(0) = \kappa(0)$, *and*

$$\|K - \kappa\|_{C^1(\mathbf{R}^n)} < \varepsilon \tag{1.6}$$

such that (1.5) has a C^2 *positive solution* $u(x)$ *satisfying*

$$u(x) = O(|x|^{2-n}) \quad \textit{as} \quad |x| \to \infty \tag{1.7}$$

and

$$u(x) \neq O(\varphi(|x|)) \quad \textit{as} \quad |x| \to 0^+. \tag{1.8}$$

Theorem 2 is stronger than Theorem 1 because the function $\kappa \colon \mathbf{R}^n \to \mathbf{R}$ in Theorem 2 does not necessarily come from a function $\kappa \in C^1(S^n)$ via the stereographic projection.

We will prove Theorem 2 in the next section.

2 Proof of Theorem 2

For our proof of Theorem 2 we will need the following simple lemma.

Lemma 1. *Suppose* $\lambda > 1$, $\{a_i\}_{i=1}^N \subset (0,\infty)$, *and* $a_1 \geq a_i$ *for* $2 \leq i \leq N$. *Then*

$$\frac{\sum_{i=1}^N a_i^\lambda}{\left(\sum_{i=1}^N a_i\right)^\lambda} \leq \frac{1 + \frac{a_2}{a_1}}{1 + \lambda\frac{a_2}{a_1}} < 1.$$

Proof. Using the hypothesises of the lemma we have

$$\frac{\sum_{i=1}^N a_i^\lambda}{\left(\sum_{i=1}^N a_i\right)^\lambda} = \frac{1+\sum_{i=2}^N \left(\frac{a_i}{a_1}\right)^\lambda}{\left(1+\sum_{i=2}^N \frac{a_i}{a_1}\right)^\lambda} \leq \frac{1+\sum_{i=2}^N \frac{a_i}{a_1}}{1+\lambda\left(\sum_{i=2}^N \frac{a_i}{a_1}\right)} \leq \frac{1+\frac{a_2}{a_1}}{1+\lambda\left(\frac{a_2}{a_1}\right)} < 1.$$

□

Proof of Theorem 2. We can assume $0 < \varepsilon < 1$, and by scaling (1.5), we can assume $\kappa(0) = 1$. Since $\nabla\kappa(0) = 0$, there exits a C^1 positive function $\hat{\kappa}\colon \mathbf{R}^n \to \mathbf{R}$ such that $\hat{\kappa}(x) \equiv 1$ in some neighborhood of the origin, $\hat{\kappa}(x) = \kappa(x)$ for $|x| \geq \varepsilon$, and $\|\hat{\kappa} - \kappa\|_{C^1(\mathbf{R}^n)} < \varepsilon/2$. Hence we can assume $\kappa \equiv 1$ in $B_\delta(0)$ for some $\delta \in (0, \varepsilon)$. Let

$$a = \frac{1}{2}\inf_{\mathbf{R}^n} \kappa \qquad \text{and} \qquad b = \sup_{\mathbf{R}^n} \kappa. \tag{2.1}$$

Let

$$w(r,\sigma) = \frac{[n(n-2)]^{\frac{n-2}{4}}\sigma^{\frac{n-2}{2}}}{(\sigma^2+r^2)^{\frac{n-2}{2}}}.$$

It is well-known that the function $V(x) = w(|x|,\sigma)$, which is sometimes called a bubble, satisfies $-\Delta V = V^{n^*}$ in $\mathbf{R}^n$ for each positive constant σ. Thus letting

$$\nu(x) = w(|x|,1)/(2b)^{n/2}$$

we have

$$-\Delta\nu = (2b)^{n^*+1}\nu^{n^*} \quad \text{in} \quad \mathbf{R}^n. \tag{2.2}$$

As $\sigma \to 0^+$, $w(|x|,\sigma)$ and each of its partial derivatives with respect to the components of x converge uniformly to zero on each closed subset of $\mathbf{R}^n - \{0\}$ and $w(0,\sigma)$ tends to ∞.

Before continuing with the proof of Theorem 2, we roughly explain the idea behind it. If $u_i(x) = w(|x-x_i|,\sigma_i)$, where $\{x_i\}_{i=1}^\infty$ is a sequence of distinct points in $B_\delta(0) - \{0\}$ which tends to the origin and $\{\sigma_i\}_{i=1}^\infty$ is a sequence of positive numbers which tends sufficiently fast to zero, then the function $\hat{u} := \sum_{i=1}^\infty u_i$ will be C^∞ in $\mathbf{R}^n - \{0\}$, will satisfy $\hat{u}(x) \neq O(\varphi(|x|))$ as $|x| \to 0^+$, and will approximately satisfy

$$-\Delta\hat{u} = \kappa\hat{u}^{n^*} = \hat{u}^{n^*} \quad \text{in} \quad B_\delta(0) - \{0\}.$$

We will find a positive bounded function $u_0\colon (\mathbf{R}^n - \{0\}) \to \mathbf{R}$ such that

$$u := u_0 + \hat{u} \qquad \text{and} \qquad K := \frac{-\Delta u}{u^{n^*}} \tag{2.3}$$

satisfy the conclusion of Theorem 2. The function u_0 will be obtained as a solution of

$$-\Delta u_0 = H(x,u_0) \qquad \text{in} \qquad \mathbf{R}^n - \{0\} \tag{2.4}$$

for some appropriate function $H\colon \mathbf{R}^n \times [0,\infty) \to \mathbf{R}$. We will use the method of sub and super-solutions to solve (2.4), using the identically zero function as a sub-solution. Thus we require that H be nonnegative.

Also, in order to force K equal to κ for $|x| \geq \delta$ and force K close to κ (at least in the C^0 norm), for $0 < |x| < \delta$, we will require that K satisfy

$$k \leq K \leq \kappa \qquad \text{in} \qquad \mathbf{R}^n - \{0\} \tag{2.5}$$

for some function $k \in C^1(\mathbf{R}^n)$ which is equal to κ for $|x| \ge \delta$ and close to κ for $|x| < \delta$. Since $-\Delta u_i = u_i^{n^*}$, it follows from (2.3) and (2.4) that (2.5) holds if and only if

$$\underline{H}(x, u_0(x)) \le H(x, u_0(x)) \le \bar{H}(x, u_0(x)) \quad \text{for} \quad x \in \mathbf{R}^n - \{0\},$$

where $\underline{H}, \bar{H} \colon \mathbf{R}^n \times [0, \infty) \to \mathbf{R}$ are defined by

$$\underline{H}(x, v) = k(x)\left(v + \sum_{i=1}^{\infty} u_i(x)\right)^{n^*} - \sum_{i=1}^{\infty} u_i(x)^{n^*},$$
$$\bar{H}(x, v) = \kappa(x)\left(v + \sum_{i=1}^{\infty} u_i(x)\right)^{n^*} - \sum_{i=1}^{\infty} u_i(x)^{n^*}.$$

Thus the nonnegative function H in (2.4) will be chosen such that $\underline{H} \le H \le \bar{H}$. After obtaining a solution u_0 of (2.4), we check at the end of the proof that K as defined by (2.3) is C^1 in $\mathbf{R}^n$. Only then does it become clear why we need $n \ge 6$. For everything to work out right, the sequences x_i and σ_i must be chosen very carefully, and a large part of the proof is devoted to explaining how this choice is made.

We now continue with the proof of Theorem 2. Elementary calculations establish the existence of numbers δ_1 and δ_2 satisfying

$$0 < 2\delta_2 < \delta_1 < \delta/2 \tag{2.6}$$

such that

$$\frac{1}{2} < \frac{w(|x - x_1|, \sigma)}{w(|x - x_2|, \sigma)} < 2 \quad \text{when } |x_1| = |x_2| = \delta_1,\ 0 < \sigma \le \delta_2,\ \text{and either } |x| \le \delta_2 \text{ or } |x| \ge \delta. \tag{2.7}$$

Let $i_0 = i_0(n, a)$ be the smallest integer greater than 2 such that

$$i_0^{n^*-1} > \frac{2^{2n^*+1}}{(2a)^{\frac{n^*}{n^*-1}}}. \tag{2.8}$$

Choose a sequence $\{x_i\}_{i=1}^{\infty}$ of distinct points in $\mathbf{R}^n$ and a sequence $\{r_i\}_{i=1}^{\infty}$ of positive numbers such that

$$|x_1| = |x_2| = \cdots = |x_{i_0}| = \delta_1, \qquad r_1 = r_2 = \cdots = r_{i_0} = \delta_2/2 < \delta_1/4, \tag{2.9}$$
$$B_{4r_i}(x_i) \subset B_{\delta_2}(0) - \{0\} \quad \text{for} \quad i > i_0, \tag{2.10}$$
$$\lim_{i \to \infty} |x_i| = 0, \tag{2.11}$$

and

$$\overline{B_{2r_i}(x_i)} \cap \overline{B_{2r_j}(x_j)} = \emptyset \quad \text{for} \quad j > i > i_0. \tag{2.12}$$

In addition to $(2.9)_1$, we require that the union of the line segments $\overline{x_1 x_2}, \overline{x_2 x_3}, \ldots, \overline{x_{i_0-1} x_{i_0}}$, $\overline{x_{i_0} x_1}$ be a regular polygon. Later we will prescribe the perimeter of this polygon.

It follows from (2.6) and (2.9) that

$$\overline{B_{2r_i}(x_i)} \subset B_{2\delta_1}(0) - \overline{B_{\delta_2}(0)} \qquad \text{for} \qquad 1 \le i \le i_0,$$

and hence by (2.10),

$$\overline{B_{2r_i}(x_i)} \cap \overline{B_{2r_j}(x_j)} = \emptyset \qquad \text{for} \qquad 1 \le i \le i_0 < j. \tag{2.13}$$

Choose a sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ of positive numbers such that

$$\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_{i_0} \quad \text{and} \quad \varepsilon_i \le 2^{-i} \quad \text{for} \quad i \ge 1. \tag{2.14}$$

Define three functions $f\colon [0,\infty)\times(0,\infty)\times(0,\infty)\to \mathbf{R}$ and $M, Z\colon (0,1)\times(0,\infty)\to(0,\infty)$ by

$$f(z,\psi,\zeta) = \psi(\zeta+z)^{n^*} - z^{n^*}, \quad M(\psi,\zeta) = \frac{\psi\zeta^{n^*}}{\left(1-\psi^{\frac{1}{n^*-1}}\right)^{n^*-1}}, \quad \text{and} \quad Z(\psi,\zeta) = \frac{\zeta\psi^{\frac{1}{n^*-1}}}{1-\psi^{\frac{1}{n^*-1}}}.$$

For each fixed $(\psi,\zeta)\in(0,1)\times(0,\infty)$, the function $f(\cdot,\psi,\zeta)\colon\ [0,\infty)\to\mathbf{R}$ assumes its maximum value $M(\psi,\zeta)$ when $z = Z(\psi,\zeta)$. Also, $f(\cdot,\psi,\zeta)$ is strictly increasing on the interval $[0, Z(\psi,\zeta)]$, and strictly decreasing on the interval $[Z(\psi,\zeta),\infty)$. Define $\hat{f}\colon\ [0,\infty)\times(0,\infty)\times(0,\infty)\to(0,\infty)$ by

$$\hat{f}(z,\psi,\zeta) = \begin{cases} f(z,\psi,\zeta), & \text{if } \psi\ge 1\\ f(z,\psi,\zeta), & \text{if } 0<\psi<1 \text{ and } 0\le z\le Z(\psi,\zeta)\\ M(\psi,\zeta), & \text{if } 0<\psi<1 \text{ and } z\ge Z(\psi,\zeta). \end{cases}$$

Then f and $\hat{f}$ are C^1, $f\le\hat{f}$, and $\hat{f}$ is non-decreasing in z, ψ and ζ.

Let N be the Newtonian potential operator over $\mathbf{R}^n$ defined by

$$(Ng)(x) = \frac{1}{(n-2)n\omega_n}\int_{\mathbf{R}^n}\frac{g(y)}{|x-y|^{n-2}}\,dy$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$.

We now introduce four sequences of real numbers

$$k_i \in \left(\frac{1}{2},1\right), \quad M_i > 3^i, \quad \rho_i\in(0,r_i), \quad \text{and} \quad \sigma_i\in(0,\delta_2), \quad i=1,2,\ldots \tag{2.15}$$

which will *always* be related as follows:

$$M_i = \frac{M(k_i,2\nu(0))}{(2\nu(0))^{n^*}} = \frac{k_i}{\left(1-k_i^{\frac{1}{n^*-1}}\right)^{n^*-1}} \tag{2.16}$$

$$\rho_i = \sup\left\{\rho>0 : N\left(\chi_{B_{2\rho}(x_i)}\right) \le \frac{\nu}{2^{i+1}(2\nu(0))^{n^*}M_i}\right\} \tag{2.17}$$

$$\sigma_i = \sup\left\{\sigma>0 : w(|x-x_i|,\sigma)\le \varepsilon_i a^{\frac{1}{n^*-1}}\nu(x) \quad \text{for} \quad |x-x_i|>\rho_i\right\} \tag{2.18}$$

where $\chi_{B_{2\rho}(x_i)}$ is the characteristic function of $B_{2\rho}(x_i)$. We also *always* assume that $k_1 = k_2 = \cdots = k_{i_0}$ and therefore the other three sequences will also *always* be constant for $1\le i\le i_0$ by $(2.9)_1$, $(2.14)_1$, and the fact that ρ_i and σ_i do not change as x_i moves on the sphere $|x| = \delta_1$.

Clearly there exist such sequences, and in what follows, we will repeatedly decrease σ_i for certain values of i while holding ε_i fixed. Because we *always* require (2.16), (2.17), and (2.18) to hold, this process of decreasing σ_i will cause ρ_i to decrease and cause M_i and k_i to increase. Nothing else will change when $i>i_0$. However, when performing this process of decreasing σ_i, $i=1,2,\ldots,i_0$ (recall that we always assume $\sigma_1=\sigma_2=\cdots=\sigma_{i_0}$ and $\rho_1=\rho_2=\cdots=\rho_{i_0}$), we will change the location of the points $x_1,x_2,\ldots,x_{i_0}$ as follows: The distance δ_1 of the points $x_1,x_2,\ldots,x_{i_0}$ from the origin (see (2.9)) will not change but they will become more bunched together because we will *always* require that the union of the line segments $\overline{x_1x_2}$, $\overline{x_2x_3},\ldots,\overline{x_{i_0-1}x_{i_0}}$, $\overline{x_{i_0}x_1}$ be a regular

i_0-gon with side length $4\rho_1$. Thus the pairwise disjoint balls $B_{2\rho_i}(x_i)$, $i = 1, 2, \ldots, i_0$, are like beads on a bracelet and decreasing σ_i, $i = 1, 2, \ldots, i_0$, causes the circumference of the bracelet, and the congruent beads on it, to get smaller. In particular,

$$\operatorname{dist}(B_i, B_j) \geq \rho_i + \rho_j \tag{2.19}$$

for $1 \leq i < j \leq i_0$ where $B_j = B_{\rho_j}(x_j)$. Hence by (2.12), (2.13), and $(2.15)_3$, inequality (2.19) holds for $1 \leq i < j$. Also, it is easy to check that

$$\min_{x \in B_j} \frac{w(|x - x_{j+1}|, \sigma_{j+1})}{w(|x - x_{j-1}|, \sigma_{j-1})} > \left(\frac{1}{3}\right)^{n-2} \qquad \text{for} \qquad 2 \leq j \leq i_0 - 1 \tag{2.20}$$

and that a similar inequality holds when j is 1 or i_0.

It follows from (2.16), (2.17), and (2.18) that for $i \geq 1$ we have

$$1 - k_i \sim \frac{1}{M_i^{\frac{1}{n^*-1}}}, \qquad M_i \sim \frac{1}{2^i \rho_i^2}, \qquad \text{and} \qquad \varepsilon_i^{\frac{2}{n-2}} \rho_i^2 \sim \sigma_i. \tag{2.21}$$

(If $\mathcal{S}$ is a finite or infinite set of positive integers, then by the statement $\alpha_i \sim \beta_i$ for $i \in \mathcal{S}$ we mean the sequence $\{\frac{\alpha_i}{\beta_i}\}_{i \in \mathcal{S}}$ is bounded between positive constants which depend at most on n, a, and b, where a and b are defined by (2.1).)

By sufficiently decreasing each term of the sequence σ_i (or equivalently by sufficiently increasing each term of the sequence M_i or k_i), we can assume that

$$\sigma_i < \left(\frac{\varepsilon_i^{\frac{2}{n-2}}}{2^i}\right)^{\frac{1}{\alpha}}, \quad \frac{1}{M_i^{\frac{\alpha}{n^*-1}}} < \varepsilon_i, \quad k_i^{\frac{n^*}{n^*-1}} > \frac{1 + \left(\frac{1}{3}\right)^{n-2}}{1 + n^* \left(\frac{1}{3}\right)^{n-2}}, \quad M_i^\alpha > 2^i, \quad \text{for } i \geq 1, \tag{2.22}$$

where $\alpha \in (0, 1/2)$ is an absolute constant to be specified later. (Actually, we will eventually take $\alpha = 1/8$, but it makes things clearer to just call it α for now.)

By (2.21) and $(2.22)_2$ we have for $1 \leq j \leq i_0$ that

$$\begin{aligned}
&\min_{x \in B_{2\rho_j}(x_j)} Z\left(k_j^{\frac{n^*}{n^*-1}}, \sum_{i=1, i \neq j}^{i_0} w(|x - x_i|, \sigma_i)\right) \\
&\quad = \min_{x \in B_{2\rho_1}(x_1)} Z\left(k_1^{\frac{n^*}{n^*-1}}, \sum_{i=2}^{i_0} w(|x - x_i|, \sigma_i)\right) \geq \min_{x \in B_{2\rho_1}(x_1)} Z\left(k_1^{\frac{n^*}{n^*-1}}, w(|x - x_2|, \sigma_2)\right) \\
&\quad \geq Z\left(k_1^{\frac{n^*}{n^*-1}}, w(6\rho_2, \sigma_2)\right) \sim \frac{1}{1 - k_1}\left(\frac{\sigma_1}{(6\rho_1)^2 + \sigma_1^2}\right)^{\frac{n-2}{2}} \sim \frac{1}{1 - k_1}\left(\frac{\sigma_1}{\rho_1^2}\right)^{\frac{n-2}{2}} \\
&\quad \sim \frac{\varepsilon_1}{1 - k_1} \geq \frac{1}{(1 - k_1) M_1^{\frac{\alpha}{n^*-1}}} \sim M_1^{\frac{1-\alpha}{n^*-1}} = M_j^{\frac{1-\alpha}{n^*-1}}.
\end{aligned} \tag{2.23}$$

Thus by sufficiently decreasing each of the equal numbers $\sigma_1, \ldots, \sigma_{i_0}$ (or equivalently by sufficiently increasing each of the equal numbers $M_1, \ldots, M_{i_0}$), we obtain

$$\min_{x \in B_{2\rho_j}(x_j)} Z\left(k_j^{\frac{n^*}{n^*-1}}, \sum_{i=1, i \neq j}^{i_0} w(|x - x_i|, \sigma_i)\right) > \nu(0) \qquad \text{for} \qquad 1 \leq j \leq i_0. \tag{2.24}$$

Also, by (2.21) we have

$$Z\left(k_j^{\frac{n^*}{n^*-1}}, \frac{1}{2M_j^{\frac{\alpha}{n^*-1}}}\right) \sim \frac{1}{1-k_j}\frac{1}{M_j^{\frac{\alpha}{n^*-1}}} \sim M_j^{\frac{1-\alpha}{n^*-1}} \qquad \text{for} \qquad j \geq 1. \tag{2.25}$$

Hence, by sufficiently decreasing each term of the sequence σ_j (or equivalently by sufficiently increasing each term of the sequence M_j), we can assume

$$Z\left(k_j^{\frac{n^*}{n^*-1}}, \frac{1}{2M_j^{\frac{\alpha}{n^*-1}}}\right) > \nu(0) \qquad \text{for} \qquad j \geq 1,$$

and therefore for $j \geq 1$ and $|x - x_j| \geq \rho_j$ we have by (2.18) that

$$\begin{aligned} w(|x-x_j|, \sigma_j) &\leq w(\rho_j, \sigma_j) \leq \varepsilon_j a^{\frac{1}{n^*-1}} \nu(0) \\ &< \nu(0) < Z\left(k_j^{\frac{n^*}{n^*-1}}, \frac{1}{2M_j^{\frac{\alpha}{n^*-1}}}\right). \end{aligned} \tag{2.26}$$

It follows from (2.21) that

$$\max_{s \geq \rho_j} \left| \frac{d}{ds}(w(s, \sigma_j)) \right| \sim \varepsilon_j 2^{\frac{j}{2}} M_j^{\frac{1}{2}} < M_j^{\frac{1}{2}} \qquad \text{for} \qquad j \geq 1 \tag{2.27}$$

by $(2.14)_2$.

We obtain from (2.21) that

$$\frac{1-k_i}{\rho_i} \sim \frac{2^{i/2}}{M_i^{\frac{n-4}{4}}} \to 0 \qquad \text{as} \qquad i \to \infty \tag{2.28}$$

because $n \geq 6$ and $M_i > 3^i$.

Let $\psi\colon\ [0, \infty) \to [0, 1]$ be a C^∞ cut-off function satisfying $\psi(t) = 1$ for $0 \leq t \leq 1$ and $\psi(t) = 0$ for $t \geq 3/2$. Define

$$k(x) = \kappa(x) + \sum_{i=1}^{\infty}(k_i - \kappa(x))\psi_i(x) \tag{2.29}$$

where $\psi_i(x) = \psi\left(\frac{|x-x_i|}{\rho_i}\right)$. Since the functions ψ_i have disjoint supports contained in $B_{2\delta_1}(0) - \{0\}$, it follows that k is well defined and finite for each $x \in \mathbf{R}^n$, $k(0) = \kappa(0) = 1$, and $k(x) = \kappa(x)$ for $|x| \geq 2\delta_1$. By $(2.15)_1$ and (2.1) we have

$$\inf_{\mathbf{R}^n} k > a. \tag{2.30}$$

Since

$$\nabla k(x) = \sum_{i=1}^{\infty} \frac{(k_i - 1)}{\rho_i}\psi'\left(\frac{|x-x_i|}{\rho_i}\right)\frac{x - x_i}{|x - x_i|} \qquad \text{for} \qquad 0 < |x| < \delta, \tag{2.31}$$

it follows from (2.28) that $k \in C^1(\mathbf{R}^n)$ and $\nabla k(0) = 0$.

Letting $u_i(x) = w(|x - x_i|, \sigma_i)$, we obtain from (2.18) and $(2.14)_2$ that

$$u_i \leq \varepsilon_i a^{\frac{1}{n^*-1}} \nu \qquad \text{in} \qquad \mathbf{R}^n - B_i, \quad i \geq 1, \tag{2.32}$$

and

$$\sum_{i=1}^{\infty} u_i \le a^{\frac{1}{n^*-1}} \nu \qquad \text{in} \qquad \mathbf{R}^n - \bigcup_{i=1}^{\infty} B_i. \tag{2.33}$$

Furthermore, by sufficiently decreasing each term of the sequence $\{\sigma_i\}_{i=1}^{\infty}$ and being mindful of the remark after equation (2.2), we can force the functions u_i to satisfy

$$u_i(x_i) > i\varphi(|x_i|) \quad \text{for} \quad i \ge 1, \tag{2.34}$$

$$\sum_{i=1}^{\infty} u_i \in C^{\infty}(\mathbf{R}^n - \{0\}),$$

$$-\Delta\left(\sum_{i=1}^{\infty} u_i\right) = \sum_{i=1}^{\infty} u_i^{n^*} \quad \text{in} \quad \mathbf{R}^n - \{0\}, \tag{2.35}$$

and $u_i + |\nabla u_i| < 2^{-i}$ in $\mathbf{R}^n - B_{2r_i}(x_i)$, $i \ge 1$. Thus by (2.12) and (2.13) we have

$$u_i + |\nabla u_i| < 2^{-i} \qquad \text{in} \qquad B_{2r_j}(x_j) \tag{2.36}$$

when $i \ne j$ and either ($j > i_0$ and $i \ge 1$) or ($1 \le j \le i_0$ and $i > i_0$). Similarly, by decreasing again each term of the subsequence $\{\sigma_i\}_{i=i_0+1}^{\infty}$ of $\{\sigma_i\}_{i=1}^{\infty}$, we can also force the functions u_i and the constants M_i to satisfy

$$\sum_{i=i_0+1}^{\infty} u_i(x) < \frac{1}{2} \min_{1 \le i \le i_0} u_i(x) \qquad \text{for} \qquad |x| \ge \delta_2, \tag{2.37}$$

$$\sum_{i=i_0+1, i\ne j}^{\infty} u_i < u_1/2 \qquad \text{in} \qquad B_{2r_j}(x_j), \quad j > i_0, \tag{2.38}$$

and

$$\frac{1}{M_j^{\frac{\alpha}{n^*-1}}} \le \min_{|x|\le\delta} u_1(x) \qquad \text{for} \quad j > i_0. \tag{2.39}$$

It follows from (2.36), (2.32), and, (2.27) that

$$\sum_{i=1, i\ne j}^{\infty} u_i + u_i^{n^*} \le C \qquad \text{in} \qquad B_j,\ j \ge 1, \tag{2.40}$$

and

$$\sum_{i=1, i\ne j}^{\infty} |\nabla u_i| + u_i^{n^*-1}|\nabla u_i| \le C M_j^{1/2} \qquad \text{in} \qquad B_j,\ j \ge 1, \tag{2.41}$$

where C is a positive constant depending at most on n, a, and b, whose value may change from line to line. (By (2.36), inequality (2.41) holds with the factor $M_j^{1/2}$ omitted, when $j > i_0$.)

By (2.17),

$$N\widehat{M} < \nu/2 \qquad \text{in} \qquad \mathbf{R}^n, \tag{2.42}$$

where

$$\widehat{M}(x) := \begin{cases} (2\nu(0))^{n^*} M_i, & \text{in } B_{\rho_i}(x_i),\ i \ge 1 \\ 0, & \text{in } \mathbf{R}^n - \bigcup_{i=1}^{\infty} B_{2\rho_i}(x_i) \\ \left(2 - \frac{|x-x_i|}{\rho_i}\right)(2\nu(0))^{n^*} M_i, & \text{in } B_{2\rho_i}(x_i) - B_{\rho_i}(x_i),\ i \ge 1. \end{cases}$$

Since $\widehat{M}$ is locally Lipschitz continuous in $\mathbf{R}^n - \{0\}$ we have $\bar{v} := \nu/(2b) + N\widehat{M} \in C^2(\mathbf{R}^n - \{0\})$ and

$$-\Delta\bar{v} = (2b)^{n^*}\nu^{n^*} + \widehat{M} \quad \text{in} \quad \mathbf{R}^n - \{0\} \tag{2.43}$$

by (2.2). It follows from (2.42) that

$$\frac{\nu}{2b} < \bar{v} < \nu \quad \text{in} \quad \mathbf{R}^n. \tag{2.44}$$

Define $\underline{H}\colon \mathbf{R}^n \times [0,\infty) \to \mathbf{R}$ by

$$\underline{H}(x,v) = k(x)\left(v + \sum_{i=1}^{\infty} u_i(x)\right)^{n^*} - \sum_{i=1}^{\infty} u_i(x)^{n^*}. \tag{2.45}$$

Then

$$\underline{H}(x,v) = f(U(x),k(x),\zeta(x,v)) \tag{2.46}$$

where

$$U(x) := \left(\sum_{i=1}^{\infty} u_i(x)^{n^*}\right)^{1/n^*} \qquad \text{and} \qquad \zeta(x,v) := v + \sum_{i=1}^{\infty} u_i(x) - U(x).$$

Define $H\colon \mathbf{R}^n \times [0,\infty) \to (0,\infty)$ by

$$H(x,v) = \hat{f}(U(x),k(x),\zeta(x,v)). \tag{2.47}$$

Then

$$H(x,v) \le M(k(x),\zeta(x,v)) = \frac{k(x)\zeta(x,v)^{n^*}}{\left(1 - k(x)^{\frac{1}{n^*-1}}\right)^{n^*-1}} \qquad \text{when} \qquad k(x) < 1 \text{ and } v \ge 0. \tag{2.48}$$

Also $H(x,v) = \underline{H}(x,v)$ if and only if either $k(x) < 1$ and $U(x) \le Z(k(x),\zeta(x,v))$ or $k(x) \ge 1$.

For $x \in \mathbf{R}^n - \bigcup_{i=1}^{\infty} B_i$ and $k(x) < 1$ we have

$$\begin{aligned} U(x) &\le \sum_{i=1}^{\infty} u_i(x) \le a^{\frac{1}{n^*-1}}\nu(x) \qquad \text{by (2.33)} \\ &\le \frac{\nu(x)k(x)^{\frac{1}{n^*-1}}}{1 - k(x)^{\frac{1}{n^*-1}}} \qquad \text{by (2.30)} \\ &\le \frac{\zeta(x,\nu(x))k(x)^{\frac{1}{n^*-1}}}{1 - k(x)^{\frac{1}{n^*-1}}} = Z(k(x),\zeta(x,\nu(x))) \end{aligned}$$

and hence

$$H(x,\nu(x)) = \underline{H}(x,\nu(x)) \qquad \text{for} \qquad x \in \mathbf{R}^n - \bigcup_{i=1}^{\infty} B_i.$$

Thus for $x \in (\mathbf{R}^n - \{0\}) - \bigcup_{i=1}^{\infty} B_i$ and $0 \le v \le \nu(x)$ we have

$$\begin{aligned} H(x,v) \le H(x,\nu(x)) = \underline{H}(x,\nu(x)) &\le k(x)\left(\nu(x) + \sum_{i=1}^{\infty} u_i(x)\right)^{n^*} \\ &\le b(2\nu(x))^{n^*} \le -\Delta\bar{v}(x), \end{aligned} \tag{2.49}$$

by (2.33) and (2.43).

Since $k(x) \equiv k_j < 1$ for $x \in B_j$, it follows from (2.48) that for $x \in B_j$ and $0 \le v \le \nu(x)$ we have

$$\begin{aligned} H(x,v) &\le \frac{k_j \zeta(x,v)^{n^*}}{\left(1 - k_j^{\frac{1}{n^*-1}}\right)^{n^*-1}} \\ &\le M_j(2\nu(x))^{n^*} \qquad \text{by (2.16) and (2.32)} \\ &\le M_j(2\nu(0))^{n^*} = \widehat{M}(x) \le -\Delta\bar{v}(x) \end{aligned} \tag{2.50}$$

by (2.43). We therefore obtain from (2.49) that

$$H(x,v) \le -\Delta\bar{v}(x) \qquad \text{for } x \in \mathbf{R}^n - \{0\} \text{ and } 0 \le v \le \nu(x).$$

Hence by (2.44), for each integer $i \ge 2$ we can use $\underline{v} \equiv 0$ and $\bar{v}$ as sub and super-solutions of the problem

$$\begin{aligned} -\Delta v = H(x,v) \quad &\text{in} \quad \frac{1}{i} < |x| < i \\ v = 0 \qquad\quad &\text{for} \quad |x| = \frac{1}{i} \quad \text{or} \quad |x| = i \end{aligned}$$

to conclude that this problem has a C^2 solution v_i satisfying $0 \le v_i \le \nu$. It follows from standard elliptic theory that some subsequence of v_i converges to a C^2 solution u_0 of

$$\left.\begin{aligned} -\Delta u_0 = H(x,u_0) \\ 0 \le u_0 \le \nu \end{aligned}\right\} \quad \text{in} \quad \mathbf{R}^n - \{0\}. \tag{2.51}$$

Define $\bar{H} \colon \mathbf{R}^n \times [0,\infty) \to (0,\infty)$ by $\bar{H}(x,v) = \hat{f}(U(x),\kappa(x),\zeta(x,v))$. Then $\underline{H} \le H \le \bar{H}$ because $k \le \kappa$. In particular,

$$\underline{H}(x,u_0(x)) \le H(x,u_0(x)) \le \bar{H}(x,u_0(x)) \qquad \text{for} \qquad x \in \mathbf{R}^n - \{0\}. \tag{2.52}$$

Since, for $|x| > \delta$,

$$\begin{aligned} U(x)^{n^*} &= \sum_{i=1}^{\infty} u_i(x)^{n^*} \\ &\le i_0 2^{n^*} u_1(x)^{n^*} + u_1(x)^{n^*} \qquad \text{by (2.7) and (2.37)} \\ &\le i_0 2^{n^*+1} u_1(x)^{n^*} = \frac{i_0^{n^*}}{i_0^{n^*-1}} 2^{n^*+1} u_1(x)^{n^*} \\ &\le \frac{(2a)^{\frac{n^*}{n^*-1}}}{2^{2n^*+1}} i_0^{n^*} 2^{n^*+1} u_1(x)^{n^*} \qquad \text{by (2.8)} \\ &\le \kappa(x)^{\frac{n^*}{n^*-1}} \left(\frac{i_0}{2} u_1(x)\right)^{n^*} \qquad \text{by (2.1)} \\ &\le \kappa(x)^{\frac{n^*}{n^*-1}} \left(\sum_{i=1}^{\infty} u_i(x)\right)^{n^*} \qquad \text{by (2.7)} \end{aligned}$$

we have for $\kappa(x) < 1$ and $v \geq 0$ that

$$U(x) \leq \frac{\left(\sum_{i=1}^{\infty} u_i(x) - U(x)\right) \kappa(x)^{\frac{1}{n^*-1}}}{1 - \kappa(x)^{\frac{1}{n^*-1}}} \leq Z(\kappa(x), \zeta(x,v)).$$

(Recall, from the first paragraph of this proof, that $\kappa(x) < 1$ implies $|x| > \delta$.) Thus, for $x \in \mathbf{R}^n$ and $v \geq 0$, we have

$$\begin{aligned} \bar{H}(x,v) &= f(U(x), \kappa(x), \zeta(x,v)) \\ &= \kappa(x) \left(v + \sum_{i=1}^{\infty} u_i(x) \right)^{n^*} - \sum_{i=1}^{\infty} u_i(x)^{n^*}, \end{aligned}$$

which together with (2.45), (2.35), (2.51), and (2.52) implies that $u := u_0 + \sum_{i=1}^{\infty} u_i$ is a C^2 positive solution of

$$k(x)u^{n^*} \leq -\Delta u \leq \kappa(x)u^{n^*} \qquad \text{in} \qquad \mathbf{R}^n - \{0\}. \tag{2.53}$$

It follows from (2.34) and (2.11) that u satisfies (1.8). We see from (2.37) and (2.51) that u satisfies (1.7).

Define $K\colon \mathbf{R}^n \to (0,\infty)$ by

$$K(x) = \frac{-\Delta u(x)}{u(x)^{n^*}} \qquad \text{for} \qquad x \in \mathbf{R}^n - \{0\} \tag{2.54}$$

and $K(0) = 1$. Then

$$K(x) = \frac{H(x, u_0(x)) + \sum\limits_{i=1}^{\infty} u_i(x)^{n^*}}{\left(u_0(x) + \sum\limits_{i=1}^{\infty} u_i(x) \right)^{n^*}} \quad \text{for} \quad x \in \mathbf{R}^n - \{0\} \tag{2.55}$$

and hence $K \in C^1(\mathbf{R}^n - \{0\})$. It follows from (2.53) and (2.54) that

$$k(x) \leq K(x) \leq \kappa(x) \quad \text{for} \quad x \in \mathbf{R}^n - \{0\}. \tag{2.56}$$

Hence, by the properties of k stated in the paragraph containing inequality (2.30), we have $K \in C^0(\mathbf{R}^n)$,

$$K(0) = k(0) = \kappa(0) = 1, \qquad \nabla K(0) = \nabla k(0) = \nabla \kappa(0) = 0, \tag{2.57}$$

and

$$K(x) = k(x) = \kappa(x) \qquad \text{for} \quad |x| \geq 2\delta_1. \tag{2.58}$$

We now show that $K \in C^1(\mathbf{R}^n)$ by showing that

$$\lim_{|x| \to 0} \nabla K(x) = 0. \tag{2.59}$$

Let $S = \{x \in \mathbf{R}^n - \{0\}\colon\ \underline{H}(x, u_0(x)) < H(x, u_0(x))\}$. It follows from (2.55) and (2.45) that

$$S = \{x \in \mathbf{R}^n - \{0\}\colon\ k(x) < K(x)\}, \tag{2.60}$$

and it follows from (2.46) and (2.47) that

$$\left.\begin{array}{l} H(x,u_0(x)) = M(k(x),\zeta_0(x)) \\ U(x) > Z(k(x),\zeta_0(x)) \end{array}\right\} \quad \text{for} \quad x \in S \tag{2.61}$$

where $\zeta_0(x) := \zeta(x,u_0(x))$. In particular, since $k(x) \ge k_j$ in $B_{2\rho_j}(x_j)$, we have

$$\begin{aligned} U(x) &> Z(k_j,\zeta_0(x)) \\ &= M_j^{\frac{1}{n^*-1}}\zeta_0(x) \quad \text{for} \quad x \in S \cap B_{2\rho_j}(x_j), \quad j \ge 1. \end{aligned} \tag{2.62}$$

We have by (2.56), (2.60), and (2.57) that

$$\nabla k(x) = \nabla K(x) \qquad \text{for} \quad x \in \mathbf{R}^n - S, \tag{2.63}$$

and thus (2.59) holds for $x \in (\mathbf{R}^n - \{0\}) - S$. We now show the limit (2.59) holds for $x \in S$. For $x \in (\mathbf{R}^n - \{0\}) - \bigcup_{i=1}^{\infty} B_{2\rho_i}(x_i)$ we have $k(x) = \kappa(x)$ and it therefore follows from (2.56) and (2.60) that $x \notin S$. Thus

$$S \subset \bigcup_{i=1}^{\infty} B_{2\rho_i}(x_i). \tag{2.64}$$

For $x \in S \cap B_{2\rho_j}(x_j)$ we have by (2.62) that

$$U(x) > \frac{k_j^{\frac{1}{n^*-1}}\left(\sum_{i=1}^{\infty} u_i(x) - U(x)\right)}{1 - k_j^{\frac{1}{n^*-1}}}$$

and thus

$$U(x) \ge k_j^{\frac{1}{n^*-1}} \sum_{i=1}^{\infty} u_i(x).$$

Hence

$$\sum_{i=1, i\neq j}^{\infty} u_i(x)^{n^*} \ge f\left(u_j(x), k_j^{\frac{n^*}{n^*-1}}, \sum_{i=1, i\neq j}^{\infty} u_i(x)\right) \qquad \text{for} \qquad x \in S \cap B_{2\rho_j}(x_j), \quad j \ge 1. \tag{2.65}$$

However, for $1 \le j \le i_0$ and $x \in B_{2\rho_j}(x_j)$ we have

$$\frac{\sum_{i=1,i\neq j}^{\infty} u_i(x)^{n^*}}{f\left(0, k_j^{\frac{n^*}{n^*-1}}, \sum_{i=1,i\neq j}^{\infty} u_i(x)\right)} = \frac{\sum_{i=1,i\neq j}^{\infty} u_i(x)^{n^*}}{k_j^{\frac{n^*}{n^*-1}}\left(\sum_{i=1,i\neq j}^{\infty} u_i(x)\right)^{n^*}} \le \frac{1 + \left(\frac{1}{3}\right)^{n-2}}{k_j^{\frac{n^*}{n^*-1}}\left(1 + n^*\left(\frac{1}{3}\right)^{n-2}\right)} < 1$$

by (2.37), Lemma 1, (2.20), and $(2.22)_3$. Thus by (2.65) and (2.24),

$$u_j(x) > Z\left(k_j^{\frac{n^*}{n^*-1}}, \sum_{i=1, i\neq j}^{\infty} u_i(x)\right) > \nu(0) \qquad \text{for} \qquad 1 \le j \le i_0 \text{ and } x \in S \cap B_{2\rho_j}(x_j). \tag{2.66}$$

Hence, by (2.26),

$$S \cap B_{2\rho_j}(x_j) = S \cap B_j \qquad \text{for} \qquad 1 \le j \le i_0, \tag{2.67}$$

and it follows from (2.23) and (2.66) that

$$u_j \geq C M_j^{\frac{1-\alpha}{n^*-1}} \qquad \text{in} \qquad S \cap B_{2\rho_j}(x_j),\ 1 \leq j \leq i_0, \tag{2.68}$$

where C is a positive constant depending at most on n, a, and b whose value may change from line to line.

Also, by (2.38), Lemma 1, and (2.7) we have for $x \in B_{2\rho_j}(x_j)$, $j > i_0$, that

$$\frac{\sum_{i=1,i\neq j}^{\infty} u_i(x)^{n^*}}{f\left(0, k_j^{\frac{n^*}{n^*-1}}, \sum_{i=1,i\neq j}^{\infty} u_i(x)\right)} = \frac{\sum_{i=1,i\neq j}^{\infty} u_i(x)^{n^*}}{k_j^{\frac{n^*}{n^*-1}} \left(\sum_{i=1,i\neq j}^{\infty} u_i(x)\right)^{n^*}} \leq \frac{1+\frac{1}{2}}{k_j^{\frac{n^*}{n^*-1}} \left(1+\frac{n^*}{2}\right)} < 1$$

by $(2.22)_3$. Thus, by (2.65) and (2.39),

$$u_j(x) > Z\left(k_j^{\frac{n^*}{n^*-1}}, \sum_{i=1,i\neq j}^{\infty} u_i(x)\right) > Z\left(k_j^{\frac{n^*}{n^*-1}}, \frac{1}{2M_j^{\frac{\alpha}{n^*-1}}}\right) \qquad \text{for} \qquad x \in S \cap B_{2\rho_j}(x_j),\ j > i_0.$$

Hence it follows from (2.26) and (2.67) that

$$S \cap B_{2\rho_j}(x_j) = S \cap B_j \quad \text{for} \quad j \geq 1, \tag{2.69}$$

and it follows from (2.25) and (2.68) that

$$u_j \geq C M_j^{\frac{1-\alpha}{n^*-1}} \quad \text{in} \quad S \cap B_{2\rho_j}(x_j), \qquad j \geq 1. \tag{2.70}$$

We see from (2.55) and (2.61) that

$$K(x) = \frac{M_j \zeta_0(x)^{n^*} + U(x)^{n^*}}{(\zeta_0(x) + U(x))^{n^*}} = \frac{M_j \left(\frac{\zeta_0(x)}{U(x)}\right)^{n^*} + 1}{\left(\frac{\zeta_0(x)}{U(x)} + 1\right)^{n^*}} \quad \text{for} \quad x \in S \cap B_j, \quad j \geq 1.$$

Thus

$$\nabla K = n^* \left(\frac{M_j \left(\frac{\zeta_0}{U}\right)^{n^*-1} - 1}{\left(\frac{\zeta_0}{U} + 1\right)^{n^*+1}}\right) \left(\nabla \frac{\zeta_0}{U}\right) \quad \text{in} \quad S \cap B_j, \quad j \geq 1,$$

and hence, by (2.62),

$$\begin{aligned} |\nabla K| &\leq n^* \left|\nabla \frac{\zeta_0}{U}\right| \\ &\leq n^* \left[\left|\nabla \frac{u_0}{U}\right| + \left|\nabla \frac{\sum_{i=1,i\neq j}^{\infty} u_i}{U}\right| + \left|\nabla \frac{u_j}{U}\right|\right] \quad \text{in} \quad S \cap B_j, \quad j \geq 1. \end{aligned} \tag{2.71}$$

We now estimate each of the three terms on the right side of (2.71). Since

$$\nabla \frac{u_j}{U} = \nabla \left(\frac{\sum\limits_{i=1}^{\infty} u_i^{n^*}}{u_j^{n^*}} \right)^{-\frac{1}{n^*}} = \nabla \left(1 + \frac{\sum\limits_{i=1, i\neq j}^{\infty} u_i^{n^*}}{u_j^{n^*}} \right)^{-\frac{1}{n^*}}$$

$$= -\frac{1}{n^*} \left(1 + \frac{\sum\limits_{i=1, i\neq j}^{\infty} u_i^{n^*}}{u_j^{n^*}} \right)^{-\frac{1}{n^*}-1} \left[\frac{\nabla \sum\limits_{i=1, i\neq j}^{\infty} u_i^{n^*}}{u_j^{n^*}} - n^* \left(\frac{\nabla u_j}{u_j^{n^*+1}} \right) \sum_{i=1, i\neq j}^{\infty} u_i^{n^*} \right],$$

it follows from (2.40), (2.41), and (2.70) that

$$\left| \nabla \frac{u_j}{U} \right| \leq C \left(\frac{M_j^{1/2}}{M_j^{\frac{(1-\alpha)n^*}{n^*-1}}} + \frac{|\nabla u_j|}{u_j^{n^*+1}} \right) \quad \text{in} \quad S \cap B_j, \quad j \geq 1. \tag{2.72}$$

Since, by (2.41) and (2.70),

$$\left| \nabla \frac{1}{U} \right| = \left| \nabla (U^{n^*})^{-\frac{1}{n^*}} \right| = \left| \frac{1}{n^*} (U^{n^*})^{-\frac{1}{n^*}-1} \nabla U^{n^*} \right| = \left| \frac{\sum\limits_{i=1, i\neq j}^{\infty} u_i^{n^*-1} \nabla u_i + u_j^{n^*-1} \nabla u_j}{U^{n^*+1}} \right|$$

$$\leq C \left(\frac{M_j^{1/2}}{M_j^{\frac{(1-\alpha)(n^*+1)}{n^*-1}}} + \frac{|\nabla u_j|}{u_j^2} \right) \quad \text{in} \quad S \cap B_j, \quad j \geq 1,$$

we have by (2.40), (2.41), (2.70), and (2.51) that

$$\left| \nabla \frac{\sum\limits_{i=1, i\neq j}^{\infty} u_i}{U} \right| \leq C \left(\left| \nabla \frac{1}{U} \right| + \frac{M_j^{1/2}}{U} \right) \leq C \left(\left| \nabla \frac{1}{U} \right| + \frac{M_j^{1/2}}{u_j} \right)$$

$$\leq C \left(\frac{M_j^{1/2}}{M_j^{\frac{1-\alpha}{n^*-1}}} + \frac{|\nabla u_j|}{u_j^2} \right) \quad \text{in} \quad S \cap B_j, \qquad j \geq 1 \tag{2.73}$$

and

$$\left| \nabla \frac{u_0}{U} \right| = \left| \frac{\nabla u_0}{U} + u_0 \nabla \frac{1}{U} \right|$$

$$\leq C \left(\frac{|\nabla u_0|}{M_j^{\frac{1-\alpha}{n^*-1}}} + \frac{M_j^{1/2}}{M_j^{\frac{(1-\alpha)(n^*+1)}{n^*-1}}} + \frac{|\nabla u_j|}{u_j^2} \right) \quad \text{in} \quad S \cap B_j, \qquad j \geq 1. \tag{2.74}$$

We now estimate ∇u_0 in B_j. Since, by (2.51), u_0 is bounded and superharmonic in $\mathbf{R}^n - \{0\}$, it is well known that

$$u_0(x) = \frac{1}{(n-2)n\omega_n} \int_{|y|<4} \frac{H(y, u_0(y))}{|x-y|^{n-2}} \, dy + h(x) \quad \text{for} \quad 0 < |x| \leq 2$$

for some continuous function $h\colon\ \overline{B_2(0)} \to \mathbf{R}$ which is harmonic in $B_2(0)$. By (2.50), (2.49), and (2.51),

$$H(x,u_0(x)) \le \begin{cases} (2\nu(0))^{n^*} M_j & \text{in } B_j \\ b(2\nu(0))^{n^*} & \text{in } (\mathbf{R}^n - \{0\}) - \bigcup\limits_{i=1}^{\infty} B_i. \end{cases}$$

It follows therefore from (2.17) and (2.51) that $|h(x)| < C$ for $|x| \le 2$. Thus $|\nabla h(x)| < C$ for $|x| \le 1$ and hence, for $x \in B_j$, we have

$$\begin{aligned} |\nabla u_0(x)| &\le \frac{1}{n\omega_n} \int_{|y|<4} \frac{H(y,u_0(y))}{|x-y|^{n-1}} dy + C \\ &\le C[I_1(x) + I_2(x) + I_3(x)] + C, \end{aligned}$$

where

$$I_1(x) = \int_{B_j} \frac{M_j}{|x-y|^{n-1}} dy \le CM_j\rho_j \le C\sqrt{M_j} \quad \text{for} \quad x \in B_j$$

by (2.21), and

$$\begin{aligned} I_2(x) &= \sum_{i=1, i\ne j}^{\infty} \int_{B_i} \frac{M_i}{|x-y|^{n-1}} dy \le C \sum_{i=1, i\ne j}^{\infty} \frac{M_i \rho_i^n}{(\operatorname{dist}(B_j, B_i))^{n-1}} \\ &\le C \sum_{i=1, i\ne j}^{\infty} \frac{\rho_i^{n-2}}{2^i(\rho_i+\rho_j)^{n-1}} \le \frac{C}{\rho_j} \sim C2^{j/2}\sqrt{M_j} \le CM_j^{\alpha+1/2} \quad \text{for} \quad x \in B_j \end{aligned}$$

by (2.21), (2.19), and $(2.22)_4$, and

$$I_3(x) = \int\limits_{B_4(0) - \bigcup\limits_{i=1}^{\infty} B_i} \frac{1}{|x-y|^{n-1}} dy \le C \quad \text{for} \quad x \in B_j.$$

Thus

$$|\nabla u_0| < CM_j^{\alpha+1/2} \qquad \text{in} \qquad B_j, \quad j \ge 1. \tag{2.75}$$

Since $n \ge 6$, we have $n^* - 1 \le 1$ and it therefore follows from (2.75) that

$$\frac{|\nabla u_0|}{M_j^{\frac{1-\alpha}{n^*-1}}} \le \frac{CM_j^{\alpha+1/2}}{M_j^{1-\alpha}} = \frac{C}{M_j^{1/2-2\alpha}} \qquad \text{in} \qquad B_j, \quad j \ge 1. \tag{2.76}$$

In order to estimate $|\nabla u_j|/u_j^2$ in $S \cap B_j$, let

$$s_j = \inf\{s > 0\colon\ S \cap B_j \subset B_s(x_j)\}$$

and $\hat{u}_j(s) = w(s, \sigma_j)$. Then $s_j \le \rho_j$ and $\hat{u}_j(s) = u_j(x)$ when $|x - x_j| = s$. Also, by (2.70) we have

$$\hat{u}_j(s) \ge CM_j^{\frac{1-\alpha}{n^*-1}} \quad \text{for} \quad 0 \le s \le s_j, \quad j \ge 1.$$

It follows therefore from (2.21) that

$$\left(\frac{\sigma_j}{\sigma_j^2 + s_j^2}\right)^2 \ge C\hat{u}_j(s_j)^{n^*-1} \ge CM_j^{1-\alpha} \ge C\left(\frac{\varepsilon_j^{\frac{2}{n-2}}}{2^j\sigma_j}\right)^{1-\alpha}$$

and thus by $(2.22)_1$ we have

$$s_j \le C \left(\frac{2^j}{\varepsilon_j^{\frac{2}{n-2}}} \right)^{\frac{1}{4}} \sigma_j^{\frac{3-\alpha}{4}} \le C \sigma_j^{\frac{3-2\alpha}{4}} \quad \text{for} \quad j \ge 1. \tag{2.77}$$

Also, for $0 \le s \le s_j$ and $j \ge 1$, we have

$$\begin{aligned} \frac{-\hat{u}_j'(s)}{\hat{u}_j(s)^2} &= \frac{(n-2)}{[n(n-2)]^{\frac{n-2}{4}}} \frac{s(\sigma_j^2 + s^2)^{\frac{n-4}{2}}}{\sigma_j^{\frac{n-2}{2}}} \\ &\le \frac{(n-2)}{[n(n-2)]^{\frac{n-2}{4}}} \frac{s_j(\sigma_j^2 + s_j^2)^{\frac{n-4}{2}}}{\sigma_j^{\frac{n-2}{2}}} \\ &\le C \frac{\sigma_j^{\frac{3-2\alpha}{4}} \left(\sigma_j^2 + \sigma_j^{\frac{3-2\alpha}{2}} \right)^{\frac{n-4}{2}}}{\sigma_j^{\frac{n-2}{2}}} \qquad \text{by (2.77)} \\ &\le C \frac{\sigma_j^{\frac{3-2\alpha}{4}} \sigma_j^{\frac{(3-2\alpha)(n-4)}{4}}}{\sigma_j^{\frac{n-2}{2}}} = C \sigma_j^{\frac{n-5-2\alpha(n-3)}{4}} \le C \sigma_j^{\frac{1-6\alpha}{4}} \end{aligned} \tag{2.78}$$

because $n \ge 6$ and $\alpha < 1/2$. Thus taking $\alpha = 1/8$, it follows from (2.76) and (2.78) that

$$\frac{|\nabla u_0|}{M_j^{\frac{1-\alpha}{n^*-1}}} \le \frac{C}{M_j^{1/4}} \qquad \text{in} \qquad B_j, \quad j \ge 1,$$

and

$$\frac{|\nabla u_j|}{u_j^2} \le C \sigma_j^{1/16} \qquad \text{in} \qquad S \cap B_j, \quad j \ge 1,$$

and hence, by (2.71), (2.72), (2.73), and (2.74), we have

$$|\nabla K| \le C \left(\frac{1}{M_j^{1/4}} + \sigma_j^{1/16} \right) \qquad \text{in} \qquad S \cap B_j, \quad j \ge 1. \tag{2.79}$$

We see therefore from (2.22), (2.69), and (2.64) that the limit (2.59) holds for $x \in S$. However, we have already shown that the limit (2.59) holds for $x \in (\mathbf{R}^n - \{0\}) - S$. Thus the limit (2.59) holds with no restriction on x, and hence $K \in C^1(\mathbf{R}^n)$.

By sufficiently decreasing σ_i for each $i \ge 1$, we can force k to satisfy

$$\|k - \kappa\|_{C^1(\mathbf{R}^n)} < \frac{\varepsilon}{4} \tag{2.80}$$

by (2.28), (2.29), (2.31), (2.57), and (2.58); and we can therefore also force K to satisfy

$$|\nabla(K - k)| = |\nabla(K - (k - \kappa))| \le |\nabla K| + \frac{\varepsilon}{4} \le \frac{\varepsilon}{2} \quad \text{in} \quad \bigcup_{j=1}^{\infty} (S \cap B_{2\rho_j}(x_j)) = S$$

by (2.79), (2.69), and (2.64). Thus by (2.63), $|\nabla(K - k)| < \frac{\varepsilon}{2}$ in $\mathbf{R}^n$. It therefore follows from (2.56) and (2.80) that K satisfies (1.6). $\square$

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