Part 1: Multiple Choice (75 points), there are 15 questions, and each question is worth 5 points.

1. $\frac{d}{dx}(x\sin(e^x)) =$

$$\frac{d}{dx}(x\sin(e^x)) = \sin(e^x) + xe^x\cos(e^x) .$$

a. is correct answer.

a. $\sin(e^x) + xe^x\cos(e^x)$
b. $\sin(e^x) + e^x\cos(e^x)$
c. $\sin(e^x) - xe^x\cos(e^x)$
d. $e^x\cos(e^x)$
e. $\sin(e^x) + x\cos(e^x)$

2. Suppose $f(x)$ is differentiable everywhere, then $\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{h} = ?$

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{h} &= \lim_{h\to 0}\frac{f(x+h)-f(x)+f(x)-f(x-h)}{h}\\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{f(x)-f(x-h)}{h}\right]\\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{f(x-h)-f(x)}{-h}\right]\\ &= f'(x)+f'(x)\end{aligned}$$

c. is the correct answer.

a. 0
b. $f'(x)$
c. $2f'(x)$
d. $f'(h)$
e. $2f'(h)$

3. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^2\frac{3}{n} =$

This is a limit of Riemann sums. Setting $x_i = \frac{i}{n}$, we see that $\Delta x_i = \frac{1}{n}$, $a = 0$, and $b = 1$. Thus,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^2\frac{3}{n} = \int_o^1 3(1+x)^2\,dx = (1+x)^3\Big|_0^1 = 8-1 = 7$$

Correct answer is e.

a. $\frac{8}{3}$
b. 8
c. 5
d. $\frac{7}{3}$
e. 7

4. $\frac{d}{dx}\left(\frac{2x-1}{\tan x}\right) =$

$$\frac{d}{dx}\left(\frac{2x-1}{\tan x}\right) = \frac{2\tan x - (2x-1)\sec^2 x}{\tan^2 x}$$

Correct answer is a.

a. $\frac{2\tan x - (2x-1)\sec^2 x}{\tan^2 x}$

b. $\frac{(2x-1)\sec^2 x - 2\tan x}{\tan^2 x}$

c. $\frac{2}{\sec^2 x}$

d. $\frac{2\sec^2 x}{\tan^2 x}$

e. $\frac{2\tan x + (2x-1)\sec^2 x}{\tan^2 x}$

5. $\int \cos(3x+1)\,dx =$

$$\int \cos(3x+1)\,dx = \frac{1}{3}\sin(3x+1) + c$$

Correct answer is d.

a. $\frac{\sin(3x+1)}{3x} + c$

b. $\sin(3x+1) + c$

c. $-\frac{\sin(3x+1)}{3} + c$

d. $\frac{\sin(3x+1)}{3} + c$

e. $3\sin(x+1) + c$

6. $\int_0^1 \frac{dx}{1+x^2} =$

$$\int_0^1 \frac{dx}{1+x^2} = \int_0^1 \frac{dx}{1+x^2} = \tan^{-1}(x)\Big|_0^1 = \frac{\pi}{4}\ .$$

Correct answer is a.

a. $\frac{\pi}{4}$

b. $\ln 2$

c. $\sin^{-1} 2$

d. 2

e. $\cos^{-1} 2$

7. If $f(x) = (x^2+1)^3$, then $f'(1)$ equals?

$$f'(1) = \frac{d}{dx}(x^2+1)^3\Big|_{x=1} = 6x(x^2+1)^2\Big|_{x=1} = 6 \cdot 2^2 = 24 .$$

Correct answer is d.

a. 8
b. 12
c. 6
d. 24
e. 48

8. $\lim_{x\to\infty}(\sin(1-1/x))^x =$

$$\lim_{x\to\infty}(\sin(1-1/x))^x = \lim_{x\to\infty} e^{x\ln(\sin(1-1/x))} = e^{-\infty} = 0 .$$

Note: $\sin 1$ lies strictly between 0 and 1. Hence $\ln(\sin 1)$ is negative.
Correct answer is c.

a. Does not exist
b. e^{-1}
c. 0
d. ∞
e. 1

9. For the function $f(x) = x^4 - x^3 + 1$ with $-1 \le x \le 1$, which of the following is true?

a. f has a global minimum at $x = 1/2$ and a global maximum at $x = -1$.
b. f has a local maximum at $x = 3/4$ and a global maximum at $x = -1$.
c. f has a point of inflection at $(1, f(1))$ and a global minimum at $x = 3/4$.
d. f has a point of inflection at $(0, f(0))$ and a global minimum at $x = 3/4$.
e. f has points of inflection at $(0, f(0))$, $(1/2, f(1/2))$, and $(3/4, f(3/4))$.

$$f'(x) = 4x^3 - 3x^2 = 4x^2\left(x - \frac{3}{4}\right)$$

For $-1 \le x < 3/4$ the derivative is negative and for $x > 3/4$ the derivative is positive. So f has a local as well as a global minimum at $x = 3/4$. This rules out answer a and b.

$$f''(x) = 12x^2 - 6x = 12x\left(x - \frac{1}{2}\right)$$

The second derivative changes sign at $x = 0$ and $x = 1/2$ only. This rules out answer 3. f does not have an inflection point when $x = 1$, which means the only possible answer is d.

10. $\tan\left(\arcsin\left(\frac{4}{5}\right)\right) =$

If $\theta = \arcsin(4/5)$, then $\sin\theta = 4/5$. A right triangle with hypotenuse of length 5 and side opposite the angle θ with length 4, must have the side adjascent to θ with length. Thus, $\tan(\theta) = 4/3$.
Correct answer is b.

a. 3/5
b. 4/3
c. 4/5
d. 5/4
e. 3/4

11. If $\log_a x = 2$ and $\ln a = 3\ln 2$, then $x =$

If $\ln a = 3\ln 2$, then $a = e^{3\ln 2} == 2^3 = 8$, and $x = a^2 = 8^2 = 64$. Correct answer is e.

a. 2
b. 4
c. 8
d. 16
e. 64

12. $\frac{d}{dx}x^{\sqrt{x}} =$

$$\frac{d}{dx}x^{\sqrt{x}} = \frac{d}{dx}e^{\sqrt{x}\ln x} = \frac{1}{2}x^{\sqrt{x}-\frac{1}{2}}(\ln x + 2) = \frac{\ln x + 2}{2\sqrt{x}}x^{\sqrt{x}}\ .$$

Correct answer is b.

a. $\sqrt{x}\,x^{\sqrt{x}-1}$
b. $\left(\frac{\ln x + 2}{2\sqrt{x}}\right)x^{\sqrt{x}}$
c. $\frac{\ln x}{2\sqrt{x}}x^{\sqrt{x}}$
d. $\left(\frac{\ln x}{2} + \sqrt{x}\right)x^{\sqrt{x}}$
e. $x^{\sqrt{x}}$

13. $\lim_{x\to\infty} [\ln(x^2+3) - 2\ln(3x+1)] =$

$$\lim_{x\to\infty} [\ln(x^2+3) - 2\ln(3x+1)] = \lim_{x\to\infty}\left[\ln\left(\frac{x^2+3}{(3x+1)^2}\right)\right] =$$
$$= \lim_{x\to\infty}\left[\ln\left(\frac{1+3/x^2}{9+6/x+1/x^2}\right)\right]$$
$$= \ln(1/9) = -\ln 9$$

Correct answer is e.

a. $-\infty$
b. -5
c. ∞
d. 0
e. $-\ln 9$

14. If $f(x)$ is a differentiable function such that $f(-1) = -4$ and $f(3) = 12$, then the mean value theorem says there is a number ξ, with $-1 < \xi < 3$, such that

$$f'(\xi)\frac{f(3)-f(-1)}{3-(-1)} = \frac{12-(-4)}{4} = 4\,.$$

Correct answer is e.

a. $f'(\xi) = -\frac{3}{4}$
b. $f'(\xi) = \frac{3}{4}$
c. $f'(\xi) = -\frac{4}{3}$
d. $f'(\xi) = \frac{4}{3}$
e. $f'(\xi) = 4$

15. If $f''(x) = 6x - 4$, $f'(0) = 1$, and $f(2) = 3$, then $f(1) =$

$$f'(x) = 3x^2 - 4x + c, f'(0) = 1 \text{ implies } c = 1.$$
$$f(x) = x^2 - 2x^2 + x + c, f(2) = 3 \text{ implies } c = 1$$

Thus, $f(1) = 1 - 2 + 1 + 1 = 1$. Correct answer is d.

a. -2
b. -1
c. 0
d. 1
e. 2

16. (10) Use the definition of the derivative to calculate $f'(x)$, with $f(x) = \sqrt{1+x}$.

$$\frac{df}{dx} = \lim_{h\to 0}\frac{\sqrt{1+x+h} - \sqrt{1+x}}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{1+x+h} - \sqrt{1+x}}{h}\,\frac{\sqrt{1+x+h} + \sqrt{1+x}}{\sqrt{1+x+h} + \sqrt{1+x}}$$
$$= \lim_{h\to 0}\frac{(1+x+h)-(1+x)}{h\left(\sqrt{1+x+h} + \sqrt{1+x}\right)} = \lim_{h\to 0}\frac{1}{\sqrt{1+x+h} + \sqrt{1+x}}$$
$$= \frac{1}{2\sqrt{1+x}}\ .$$

17. (10) Use the definition of the definite integral to calculate $\int_{-1}^{2}(x^2+1)\,dx$.

$$\int_{-1}^{2}(x^2+1)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(-1+\tfrac{3i}{n}\right)^2+1\right)\tfrac{3}{n}$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{9i^2}{n^2} - 6\tfrac{i}{n} + 2\right)\tfrac{3}{n}$$
$$= \lim_{n\to\infty}\frac{3}{n}\left(\frac{9}{n^2}\,\frac{n(n+1)(2n+1)}{6} - \frac{6}{n}\,\frac{n(n+1)}{2} + 2n\right)$$
$$= 3\lim_{n\to\infty}\left(\frac{9}{6}\,\frac{n(n+1)(2n+1)}{n^3} - 3\frac{n(n+1)}{n^2} + 2\right)$$
$$= 3(3-3+2) = 6\ .$$

18. (20) Sketch the graph of the function

$$f(x) = \frac{x-1}{(x+1)^2}.$$

Be sure to indicate any asymptotic behavior, where the function is increasing, decreasing, concave up, concave down, locations of local and global extrema, and points of inflection.

Before graphing this function we make a few observations:

a. It's domain consists of all real numbers except $x = -1$. $f(0) = -1$, and $f(1) = 0$.

b. The denominator is always positive so the sign of f is determined by the sign of the numerator. In particular if $x > 1$, then $f(x) > 0$, and if $x < 1$, then $f(x) < 0$.

c. The x-axis is a horizontal asymptote as $\lim_{x \to \pm\infty} f(x) = 0$.

d. There is a vertical asymptote at $x = -1$, and $\lim_{x \to -1^+} f(x) = -\infty$, and $\lim_{x \to -1^-} f(x) = -\infty$.

Note that for x close to -1 the numerator is close to -2.

The first derivative of f is

$$\frac{df}{dx} = \frac{3-x}{(x+1)^3},$$

(a, b)	f'	f
$(-\infty, -1)$	< 0	↓
$(-1, 3)$	> 0	↑
$(3, \infty)$	< 0	↓

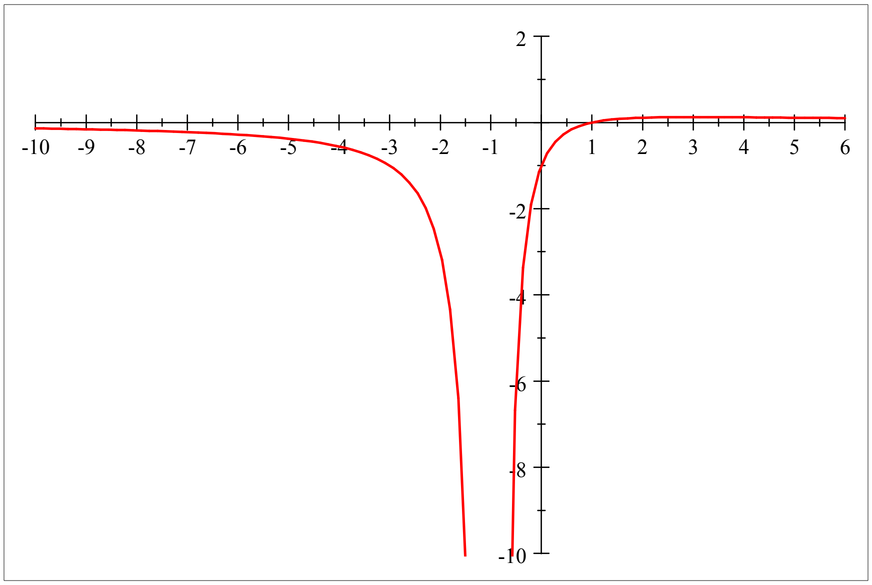
The table and the fact that f is negative for $x < -1$ tells us that f has a local and global maximum at $x = 3$, with $f(3) = \frac{1}{8}$.

The second derivative of f equals

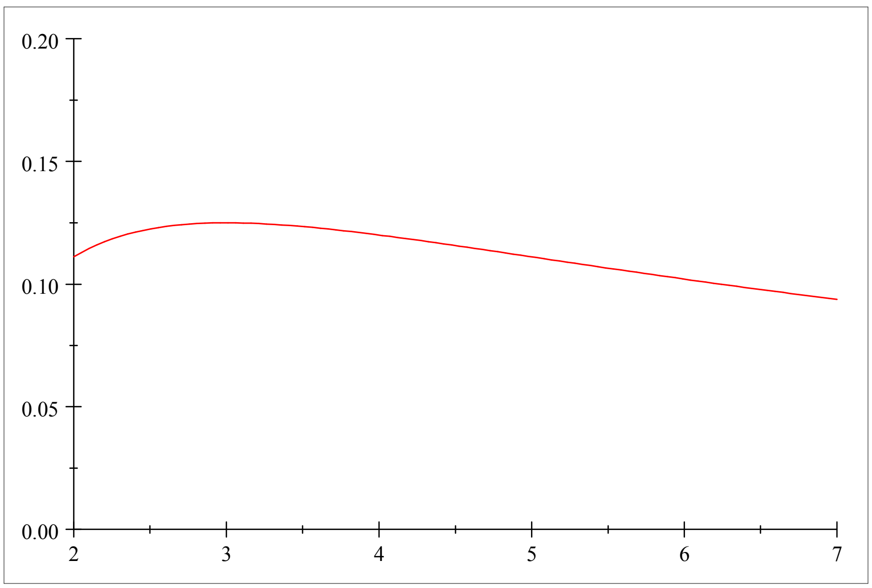
$$\frac{d^2f}{dx^2} = \frac{2(x-5)}{(x+1)^4},$$

(a, b)	f''	f
$(-\infty, -1)$	< 0	∩
$(-1, 5)$	< 0	∩
$(5, \infty)$	> 0	∪

So f is concave down on the two intervals $(-\infty, -1)$ and $(-1, 5)$, and concave up on the interval $(5, \infty)$. Thus, f has an inflection point at $(5, f(5)) = \left(5, \frac{1}{9}\right)$. A plot of f is shown below



It's not easy to see the global max and inflection point in this plot. The following one makes them slightly more visible



19. (20) A rectangular box with a square base and no top has a volume of 2000 cubic inches. If the material for the base costs 4 times as much as the material for the sides, what should the dimensions of the box be in order to minimize the cost?

If x denotes the length of one side of the box's base and h denotes its height, then

$$2000 = V = hx^2$$
$$C(x,h) = 4x^2 + 4xh\ .$$

Solving the top equation for h as a function of x and inserting this into the expression for the material cost, we have

$$C(x) = 4x^2 + 4x\left(\frac{2000}{x^2}\right)$$
$$= 4\left(x^2 + \frac{2000}{x}\right),\ 0 < x\ .$$

We note that x can, theoretically be any positive number, $\lim_{x\to 0^+} C(x) = \infty$, $\lim_{x\to\infty} C(x) = \infty$, and $C(x) > 0$ for all postive x. Since $C(x)$ is differentiable everywhere on its domain, the global minimum will also be a local minimum and must occur at a place where $C'(x) = 0$.

$$C'(x) = \frac{8}{x^2}(x^3 - 1000)\ .$$

Note that for $0 < x < 10$, $C'(x) < 0$ and for $x > 10$, $C' > 0$. Thus, the minimum cost occurs at $x = 10$ inches and $h = \frac{2000}{100} = 20$ inches, giving a box twice as tall as it is wide.