Math 151 Sections 510 and 511, Spring 2000

Solutions to Final Exam, Version A

Part 1. Multiple choice problems. Each problem is worth 5 points. Read each question carefully. No calculators are allowed on this part.

1. $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$ equals?

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{2x}{2} = 4$$

2. $\lim_{x \to \infty} \frac{x^2 - 5x + 1}{3x^2 + 4x - 8}$ equals?

$$\lim_{x \to \infty} \frac{x^2 - 5x + 1}{3x^2 + 4x - 8} = \lim_{x \to \infty} \frac{2x - 5}{6x + 4} = \lim_{x \to \infty} \frac{2}{6} = \frac{1}{3}$$

3. $\lim_{x \to 0} \frac{2x}{\sin x}$ equals?

$$\lim_{x \to 0} \frac{2x}{\sin x} = 2 \lim_{x \to 0} \frac{x}{\sin x} = 2 \cdot 1 = 2$$

4. $\lim_{x \to \infty} \frac{\ln(1 + e^x)}{5x}$ equals

$$\lim_{x \to \infty} \frac{\ln(1 + e^x)}{5x} = \lim_{x \to \infty} \frac{\frac{e^x}{1 + e^x}}{5} = \lim_{x \to \infty} \frac{e^x}{5e^x} = \frac{1}{5}$$

5. $\frac{d}{dx}\left[\frac{2x - 1}{x^2 + 1}\right]$ equals?

$$\frac{d}{dx}\left[\frac{2x - 1}{x^2 + 1}\right] = \frac{-2x^2 + 2x + 2}{(x^2 + 1)^2}$$

6. $\frac{d}{dx}\left[(2x + 1)^{29}\right]$ equals?

$$\frac{d}{dx}\left[(2x + 1)^{29}\right] = 58\,(2x + 1)^{28}$$

7. $\frac{d}{dt}\ln\left(1+t^2\right)$ equals?

$$\frac{d}{dt}\ln\left(1+t^2\right)=\frac{2t}{1+t^2}$$

8. $\frac{d}{dx}\left[e^{\sin 2x}\right]$ equals?

$$\frac{d}{dx}\left[e^{\sin 2x}\right]=2\left(\cos 2x\right)e^{\sin 2x}$$

9. $\int_0^1 x^2\,dx$ equals

$$\int_0^1 x^2\,dx=\left.\frac{x^3}{3}\right|_0^1=\frac{1}{3}$$

10. $\int_0^1 \frac{dx}{1+x^2}$ equals?

$$\int_0^1 \frac{dx}{1+x^2}=\arctan x|_0^1=\frac{1}{4}\pi$$

11. Find an equation for the tangent line to the curve $y=x^3-3$ at the point $(2,5)$.

The slope of the tangent line at the point $\left(x,x^3-3\right)$ equals $3x^2$. Thus, the equation of the tangent line at the point $(2,5)$ is

$$y-5=12\left(x-2\right).$$

12. If $f(x) = (1+x)^{1/3}$, then for x close to 0, the best linear approximation to $f(x)$ equals?

The linear approximation of $f(x)$ at the point x_0 is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

The derivative of $(1+x)^{1/3}$ equals $\frac{1}{3}(1+x)^{-2/3}$ Thus, if $x_0 = 0$, we have

$$(1+x)^{1/3} \approx 1 + \frac{1}{3}x.$$

13. If the second derivative of $f(x)$ equals $x^{-5} + 2x$, then $f(x)$ could equal which of the following?

If the second derivative equals $x^{-5} + 2x$, then we must have

$$\begin{aligned} f'(x) &= \frac{-1}{4}x^{-4} + x^2 + C_1 \\ f(x) &= \frac{1}{12}x^{-3} + \frac{x^3}{3} + C_1 x + C_2 \end{aligned}$$

The only match from the possible choices is

$$\frac{x^{-3}}{12} + \frac{x^3}{3} + x.$$

14. Find $\lim_{x \to 0} (1+x)^{1/x}$.

$$\begin{aligned} \lim_{x \to 0} (1+x)^{1/x} &= \lim_{x \to 0} e^{\ln(1+x)/x} \\ &= e^{\lim_{x \to 0} \ln(1+x)/x} \\ &= e^1 = e \end{aligned}$$

15. Find a function $g(x)$ such that $g'(x) = \frac{1}{x \ln x}$.

$$g(x) = \int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln u + C = \ln(\ln x) + C$$

Part 2. Worked out problems. Show all work for full credit. You may not use your calculator on this part of the examination until all Scantron forms are collected. Each problem is worth 15 points.

16. Let C denote the curve defined by the equation $y^5 + 3x^2y^2 - x^3 = 43$. Find an equation for the tangent line to this curve at the point $(1, 2)$.

Differentiate the equation implicitly, solve for y' and evaluate it at $x = 1$, $y = 2$.

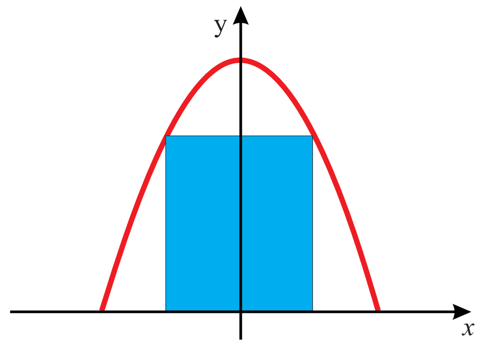
$$5y^4y' + 6xy^2 + 6x^2yy' - 3x^2 = 0$$

$$y' = \left.\frac{3x^2 - 6xy^2}{5y^4 + 6x^2y}\right|_{(1,2)} = \frac{3 - 6 \cdot 4}{5 \cdot 16 + 6 \cdot 2} = -\frac{21}{92}$$

Thus, the equation for the tangent line is

$$y - 2 = -\frac{21}{92}\,(x - 1)\,.$$

17. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 9 - x^2$. See the figure below.



The area of the inscribed rectangle equals

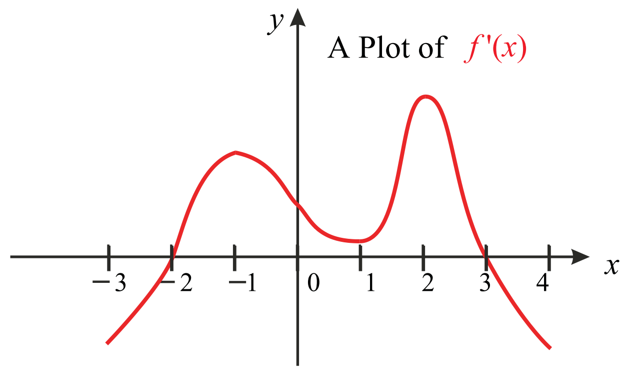
$$A(x) = 2x\left(9 - x^2\right) = 18x - 2x^3 \text{ for } 0 \le x \le 3.$$

Calculate the derivative of the area function, and set it equal to zero.

$$\begin{aligned} A'(x) &= 18 - 6x^2 = 0 \\ x^2 &= 3 \\ x &= \sqrt{3} \end{aligned}$$

The base of the rectangle has length $2\sqrt{3}$ and the height of the rectangle is 6.

18. The following plot is the graph of the derivative of a function $f(x)$. Use this graph to answer the following questions about $f(x)$. Remember, the plot below is NOT the graph of $f(x)$, it is the graph of $f'(x)$.



(a) On what intervals is the function $f(x)$ increasing? Explain.

$f(x)$ is increasing if the derivative is positive. Thus, $f(x)$ is increasing on the interval $[-2, 3]$

(b) On what intervals is the function $f(x)$ concave down? Explain

$f(x)$ is concave down when $f'(x)$ is decreasing. Thus, $f(x)$ is concave down on the intervals $[-1, 1]$ and $[2, \infty)$.

(c) At what values of x is $f(x)$ a local minimum. Explain.

$f(x)$ has a local minimum at $x = -2$. For $x < -2$ the derivative is negative, which means that $f(x)$ is decreasing, and for $x > 2$ the derivative is positive, which means the $f(x)$ is increasing. Hence, $f(x)$ must have a local miniumum at $x = -2$. $f(x)$ has a local maximum at $x = 3$.

(d) At what values of x is the point $(x, f(x))$ an inflection point?

At $x = -1$, 1, and 2. At each of these points $f'(x)$ either changes from increasing to decreasing or vice-versa. Thus, $f(x)$ has inflection points at these three points.

19. The table below gives values of $f(x)$ at various values of x. Use a right Riemann sum with 4 equal length subintervals to estimate the value of $\int_1^3 f(x)\,dx$.

x	0	0.5	1	1.5	2	2.5	3	3.5	4
$f(x)$	-1	1	2	2.6	2.7	2	1	-1	-2

$$\int_1^3 f(x)\,dx \approx \frac{4}{4}(2+2.7+1-2) = 3.7$$

20. A. What is the limit definition of the derivative of a function $f(x)$ at a point a?

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

B. Use the definition of the derivative to find $f\,'(3)$, where $f(x)=\sqrt{x}$.

$$\begin{aligned} f\,'(3) &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{\sqrt{3+h}-\sqrt{3}}{h} \\ &= \lim_{h\to 0}\frac{\left(\sqrt{3+h}-\sqrt{3}\right)\left(\sqrt{3+h}+\sqrt{3}\right)}{h\left(\sqrt{3+h}+\sqrt{3}\right)} \\ &= \lim_{h\to 0}\frac{(3+h-3)}{h\left(\sqrt{3+h}+\sqrt{3}\right)} = \lim_{h\to 0}\frac{1}{\left(\sqrt{3+h}+\sqrt{3}\right)} \\ &= \frac{1}{2\sqrt{3}} \end{aligned}$$