1. (10) Define the following:

   (a) An upper Riemann sum for the function $f(x)$ on the interval $[2, 6]$. If you tell me what this is for just two subintervals, I'll be happy.

   Let $\{x_i\}_{i=0}^{n}$ be any partition of $[a, b]$. Then the upper Riemann sum of f for this partition is the sum

$$\sum_{i=1}^{n} f(\xi_i)\,\Delta x_i\,,$$

   where $f(\xi_i)$ is the maximum value of f on the subinterval $[x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1}$.

   (b) The definite integral of f over the interval $[a, b]$.

   The definite integral is defined as a limit of Riemann sums $\sum_{i=1}^{n} f(\xi_i)\,\Delta x_i$ as n goes to infinity. It is assumed that the $\Delta x_i = x_i - x_{i-1}$ go to zero as n goes to infinity. As usual the x_i for $0 \le i \le n$, form a partition of $[a, b]$, and the point ξ_i can be any point in the i^{th} subinterval, $[x_{i-1}, x_i]$.

   (c) $f'(5)$ for a function $f(x)$.

$$f'(5) = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h}\,.$$

   (d) State the mean value theorem.

   Let $f(x)$ be continuous on the interval $[a, b]$ and differentiable on the interval (a, b), then there is a number c, $a < c < b$, such that

$$f(b) - f(a) = f'(c)\,(b - a)$$

2. (10) Use the definition of the definite integral to calculate

$$\int_{-1}^{3} \left(-x^2 + 5\right)\, dx$$

Set $x_i = -1 + \dfrac{4i}{n}$. Then

$$\begin{aligned}
\int_{-1}^{3} \left(-x^2 + 5\right)\, dx &= \lim_{n \to \infty} \sum_{i=1}^{n} \left(-\left(-1 + \frac{4i}{n}\right)^2 + 5\right) \frac{4}{n} \\
&= \lim_{n \to \infty} \frac{4}{n} \sum_{i=1}^{n} \left(4 + \frac{8i}{n} - \frac{16i^2}{n^2}\right) \\
&= \lim_{n \to \infty} \frac{4}{n} \left\{4n + \frac{8}{n}\frac{n(n+1)}{2} - \frac{16}{n^2}\frac{n(n+1)(2n+1)}{6}\right\} \\
&= 16 + 16 - 4\frac{16}{3} = \frac{32}{3}
\end{aligned}$$

3. (10) If a force acts on an object with magnitude $F(x)$ when the object is at x, then the work done by the force on the body as it moves from $x = a$ to $x = b$ is given by

$$\text{Work } = \int_a^b F(x)\, dx\,.$$

Explain how this formula is arrived at.

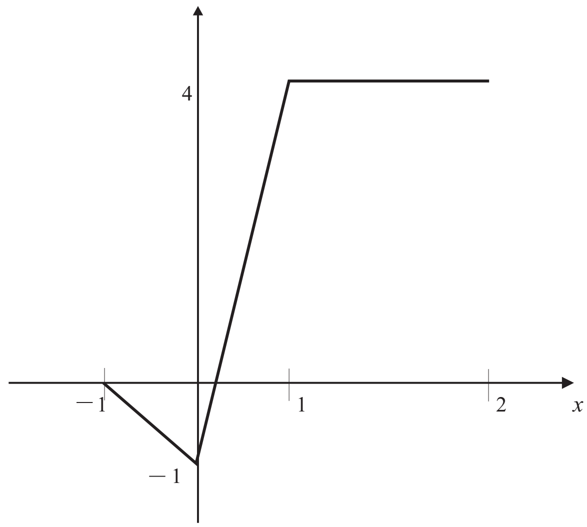
Work is force times distance. The justification for the above formula is as follows. Divide the line segment $[a, b]$ into small subintervals $[x_{i-1}, x_i]$ for $i = 1, \cdots, n$. If $\Delta x_i = x_i - x_{i-1}$ is small, we can assume that $F(x)$ is essentially constant on this subinterval with constant value $F(x_i)$. Thus, the work done on the object as it moves from x_{i-1} to x_i is approximately $F(x_i)\,\Delta x_i$. So the work done on the object as it moves from $x = a$ to $x = b$ is approximately

$$\text{Work} \approx \sum_{i=1}^{n} F(x_i)\,\Delta x_i$$

The next step is to assert that the actual work is the limit of such sums as n goes to infinity. The sums are recognized as Riemann sums so the limit will be the definite integral of F from a to b. That is,

$$\text{Work} = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i)\,\Delta x_i = \int_a^b F(x)\, dx\,.$$

4. (10) The graph of $f(x)$ is shown below. Use the graph to calculate $\int_{-1}^{2} f(x)\, dx$.



Note that the graph of f must cross the x-axis at $x = \frac{1}{5}$ since the slope of the straight line between 0 and 1 is 5. Thus,

$$\begin{aligned} \int_{-1}^{2} f(x)\, dx &= \int_{-1}^{1/5} f(x)\, dx + \int_{1/5}^{1} f(x)\, dx + \int_{1}^{2} f(x)\, dx \\ &= -\frac{1}{2}\left(\frac{6}{5}\right) + \frac{1}{2}\left(\frac{4}{5}\right)(4) + 4 \\ &= 5 \end{aligned}$$

5. (10) The function $\mathrm{Si}(x)$ is defined as

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt\ ,$$

where $\frac{\sin t}{t}$ is defined to be 1 when $t = 0$.

(a) What is the derivative of $\mathrm{Si}(x)$?

$$\frac{d}{dx}\mathrm{Si}(x) = \frac{d}{dx}\int_0^x \frac{\sin t}{t}\, dt = \frac{\sin x}{x}$$

(b) On the interval $[0, 2\pi]$, where is $\mathrm{Si}(x)$ an increasing function?

Since $\sin x$ is positive on $(0, \pi)$ and negative on $(\pi, 2\pi)$, the derivative of $\mathrm{Si}(x)$ is positive on $(0, \pi)$ and negative on $(\pi, 2\pi)$. Thus, the function $\mathrm{Si}(x)$ is increasing on $[0, \pi]$.

6. (10) Using $\ln x = \int_1^x \frac{dt}{t}$, verify that $\ln(xy) = \ln x + \ln y$.

For any value of y set $f(x) = \ln x + \ln y - \ln xy$. Then $f'(x) = \frac{1}{x} - \frac{1}{x} = 0$. Thus, f must be a constant. Evaluating f at $x = 1$, we have $f(1) = \ln 1 + \ln y - \ln y = 0$. Thus, $f(x) = 0$ for all x, and we conclude that

$$\ln x + \ln y = \ln xy$$

7. (20) Compute the following integrals:

(a) $\int_0^\pi \sin t\, dt = -\cos t|_0^\pi = (-\cos\pi) - (-\cos 0) = 1 + 1 = 2$

(b) $\int \frac{t-1}{1+t^2}\, dt = \int \frac{t}{1+t^2}\, dt - \int \frac{1}{1+t^2}\, dt = \frac{1}{2}\ln\left(1+t^2\right) - \tan^{-1} t + c$

(c) $\int \frac{x\, dx}{\sqrt{3-x^2}}$ Set $u = 3 - x^2$. Then $du = -2x\, dx$

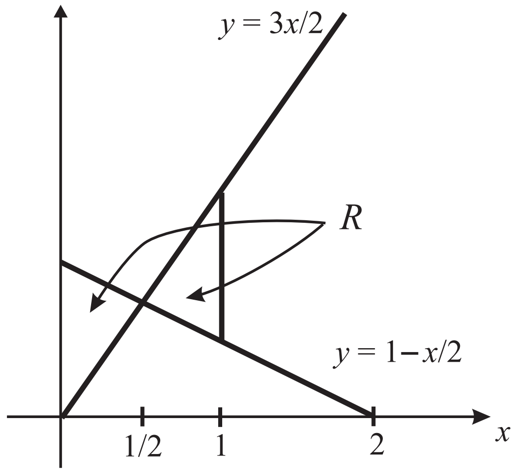
$$\begin{aligned}\int \frac{x\, dx}{\sqrt{3-x^2}}\, dx &= \frac{-1}{2}\int \frac{du}{\sqrt{u}} = -u^{1/2} + c \\ &= -\left(3-x^2\right)^{1/2} + c\end{aligned}$$

(d) $\int_0^1 xe^{3x^2}\, dx$ Set $u = 3x^2$. Then $du = 6x\, dx$.

$$\begin{aligned}\int_0^1 xe^{3x^2}\, dx &= \frac{1}{6}\int_0^3 e^u\, du = \left.\frac{e^u}{6}\right|_0^3 \\ &= \frac{e^3-1}{6}\end{aligned}$$

8. (20) Let R be the region bounded by the curves $y = \frac{3}{2}x$, $y = 1 - x/2$, $x = 0$, and $x = 1$.

The region R is shown below



(a) Find the area of the region R.

$$\begin{aligned}
\text{area} &= \int_0^1 \left| \frac{3x}{2} - \left(1 - \frac{x}{2}\right) \right| dx \\
&= \int_0^{1/2} \left[\left(1 - \frac{x}{2}\right) - \frac{3x}{2} \right] dx + \int_{1/2}^1 \left[\frac{3x}{2} - \left(1 - \frac{x}{2}\right) \right] dx \\
&= \left(x - x^2\right)\Big|_0^{1/2} + \left(x^2 - x\right)\Big|_{1/2}^1 \\
&= \left(\frac{1}{2} - \frac{1}{4}\right) + \left[(1-1) - \left(\frac{1}{4} - \frac{1}{2}\right)\right] \\
&= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}
\end{aligned}$$

(b) Find the volume of the solid obtained by rotating R about the y-axis.
Using shells is the easiest method.

$$\begin{aligned}
\text{volume} &= 2\pi \int_0^1 x \left| \frac{3x}{2} - \left(1 - \frac{x}{2}\right) \right| dx \\
&= 2\pi \int_0^{1/2} x \left[\left(1 - \frac{x}{2}\right) - \frac{3x}{2} \right] dx + 2\pi \int_{1/2}^1 x \left[\frac{3x}{2} - \left(1 - \frac{x}{2}\right) \right] dx \\
&= 2\pi \int_0^{1/2} \left(x - 2x^2\right) dx + 2\pi \int_{1/2}^1 \left(2x^2 - x\right) dx \\
&= 2\pi \left(\frac{x^2}{2} - \frac{2}{3}x^3\right)\Big|_0^{1/2} + 2\pi \left(\frac{2}{3}x^3 - \frac{x^2}{2}\right)\Big|_{1/2}^1 \\
&= 2\pi \left[\left(\frac{1}{8} - \frac{2}{24}\right) + \left(\left(\frac{2}{3} - \frac{1}{2}\right) - \left(\frac{2}{24} - \frac{1}{8}\right) \right) \right] \\
&= \frac{\pi}{2}
\end{aligned}$$