1. (5) Correctly print your name.

2. (20) If a definition is asked for, be sure that you give the definition, if you are asked to state a theorem, be sure to put in any needed hypotheses.

   (a) Define the deriviative of a function, f, at $x = 5$.

$$f'(5) = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$$

   (b) What is the ratio test?

   The ratio test is used to determine if an infinite series of postive terms converges. Consider the series $\sum_{n=0}^{\infty} a_n$, where $a_n > 0$ for each n. Let $L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$. Then if $L < 1$ the series converges, if $L > 1$ the series diverges, and if $L = 1$ the test is inconclusive.

   (c) If $f(x) = \sum_{n=0}^{\infty} a_n (x-2)^n$, is defined for $|x-2| < 1$, how are the coefficients a_n related to the function f ?

$$a_n = \frac{f^{(n)}(2)}{n!} \text{ for } n = 0,\ 1,\ \cdots$$

   (d) What is the trapezoid rule?

   The trapezoid rule is a method used to numerically approximate a definite integral. The algorithm is

$$\int_a^b f(x)\, dx \approx \frac{b-a}{2n} \left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right),$$

   where x_i are the points of a uniform partition of $[a, b]$. That is, $x_i = a + i\frac{b-a}{n}$ for $i = 0$, $1, \cdots, n$.

3. (10) Use the definition to verify that $\lim_{n \to \infty} \frac{n}{2n+1} = \frac{1}{2}$.

   Let $\epsilon > 0$. Set $N = \frac{1}{4}\left(\frac{1}{\epsilon} - 2\right)$. Then for $n > N$ we have

$$\left| \frac{n}{2n+1} - \frac{1}{2} \right| = \left| \frac{1}{4n+2} \right| < \epsilon\,.$$

4. (10) State the mean value theorem and use it to show that if the derivative of a function is zero everywhere, then the function must be a constant.

Suppose $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there is a point ξ in the open interval, $a < \xi < b$, such that

$$f(b) - f(a) = f'(\xi)(b-a) .$$

Now suppose the $f'(x) = 0$ for all x. Then for any x in $(a,b]$, we have

$$f(x) - f(a) = f'(\xi)(x-a) = 0.$$

Thus, we have $f(x) = f(a)$ for all x in $[a,b]$, or f is a constant function.

5. (25) Evaluate the following integrals.

(a) $\int \tan x \, dx$

$$\begin{aligned} \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx = -\ln(|\cos x|) + c \\ &= \ln(|\sec x|) + c \end{aligned}$$

(b) $\int \frac{dx}{2x+1}$

$$\int \frac{dx}{2x+1} = \frac{1}{2}\ln|2x+1| + c$$

(c) $\int_0^\infty xe^{-x} \, dx$

Use integration by parts.

$$\begin{aligned} \int_0^\infty xe^{-x} \, dx &= -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} \, dx \\ &= -e^{-x}\Big|_0^\infty = 1 \end{aligned}$$

(d) $\int \frac{dx}{(x-1)(x^2+1)}$

Use partial fractions.

$$\begin{aligned} \frac{1}{(x-1)(x^2+1)} &= \frac{a}{x-1} + \frac{bx+c}{x^2+1} \\ 1 &= a\left(x^2+1\right) + (x-1)(bx+c) \end{aligned}$$

This last equation leads to $a = 1/2$, and $b = c = -1/2$

$$\begin{aligned}\int \frac{dx}{(x-1)(x^2+1)} &= \int \left(\frac{1}{2(x-1)} - \frac{1}{2} \frac{x+1}{x^2+1} \right) dx \\ &= \frac{1}{2} \ln(x-1) - \frac{1}{4} \ln\left(x^2+1\right) - \frac{1}{2} \arctan x + c\end{aligned}$$

(e) $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$

Use the trig substitution $x = \sec\theta$, $dx = \sec\theta\tan\theta$

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2-1}} &= \int \frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta-1}} d\theta \\ &= \int \sec\theta\, d\theta = \ln(|\sec\theta + \tan\theta|) + c \\ &= \ln\left(\left|x + \sqrt{x^2-1}\right|\right) + c\end{aligned}$$

6. (20) Let R be the region bounded by the curves $y = \sin x$, $y = e^x$, $x = 0$, and $x = \pi$.

(a) Set up, but do not evaluate, an integral that gives the area of R.

$$\text{area} = \int_0^\pi \left(e^x - \sin x\right) dx \approx 20.141$$

(b) Set up, but do not evaluate, an integral that gives the volume of the solid of revolution obtained by rotating R about the y-axis.

$$\text{volume} = 2\pi \int_0^\pi x\left(e^x - \sin x\right) dx \approx 297.93$$

7. (10) Determine whether the series below converge absolutely, or conditionally, or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi/6)}{n\sqrt{n}}$

Each of the terms of this series is positive, and we also have

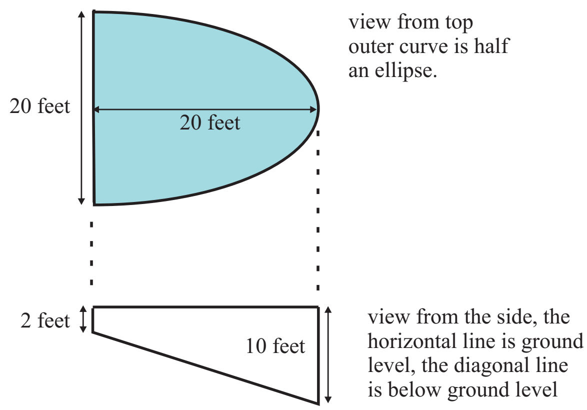
$$\frac{\cos(n\pi/6)}{n\sqrt{n}} \leq \frac{1}{n\sqrt{n}} \text{ for each } n.$$

Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges (p-series with $p = 3/2$), the original series also converges by the comparison test. Since all of the terms are positive the series also converges absolutely.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$

The terms $a_n = \frac{1}{2n}$ are positive and monotone decreasing to 0 as n goes to infinity. Thus, by the alternating series test the series converges. It does not converge absolutely as the series $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges by comparison to the harmonic series.

8. (30) I'm building a water garden in my back yard. The shape of the garden and its dimensions are given below. The right half of the outer perimeter is part of an ellipse whose equation is $\frac{x^2}{400} + \frac{y^2}{100} = 1$, and a vertical cross sectional slice that is perpendicular to the major axis of the ellipse is a rectangle.



(a) What is the length of the perimeter of the water garden. Your answer will probably include an integral. You do not have to evaluate the integral. Just be sure that you explain what it represents.

$$
\begin{aligned}
\text{length} &= 20 + 2\int_0^{20} \sqrt{1 + \frac{x^2}{1600 - 4x^2}}\, dx \\
&\approx 68.442.
\end{aligned}
$$

The value 20 is the length of vertical straight part. The integral comes from parametrizing the top eliptic part of the pond as $\Gamma(x) = \left(x, 10\sqrt{1 - \frac{x^2}{400}}\right)$ for $0 \le x \le 20$. Since the length of the elliptic part below the x-axis is the same as the length above the axis, twice the integral (which is the length of the upper part of the eliptic curve) represents the length of the eliptic part of the curve. The integrand comes from the arclength formula

$$
\int_a^b \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\, dx = \int_0^{20} \sqrt{\frac{x^2}{4\left(400 - x^2\right)} + 1}\, dx
$$

(b) What is the volume of water that my garden will hold. Your answer will probably include an integral. You do not have to evaluate the integral. Just be sure that you explain what it represents.

$$\begin{aligned} \text{volume} &= 2\int_0^{20} \left(\frac{2x}{5}+2\right)\sqrt{100-\frac{x^2}{4}}\,dx \\ &\approx 1695.0 \end{aligned}$$

The integrand consists of two pieces $\left(\frac{2x}{5}+2\right)$, which is the depth of water in the pool at x units from the straight line edge of the pool, and $2\sqrt{100-\frac{x^2}{4}}$, which is the distance between the two points on the perimeter of the pool with abscissa x. The product of these two terms gives the cross sectional area of the pool at the point x.

9. (20) Let $f(x) = \ln(x+1)$.

(a) Compute the n^{th} order Taylor polynomial expansion of f about the point $x = 0$.

The first step is to calculate the n^{th} derivative of f and evaluate it at $x = 0$. The table below gives these expressions

n	$f^{(n)}$	$f^{(n)}(0)$
0	$\ln(x+1)$	0
1	$\frac{1}{x+1}$	1
2	$\frac{-1}{(1+x)^2}$	-1
3	$\frac{2}{(1+x)^3}$	2
4	$\frac{-3!}{(1+x)^4}$	$-3!$
i	$\frac{(-1)^{(i-1)}(i-1)!}{(1+x)^i}$	$(-1)^{(i-1)}(i-1)!$

The n^{th} order Taylor polynomial is

$$\begin{aligned} \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!}x^i &= \sum_{i=1}^{n} \frac{(-1)^{(i-1)}(i-1)!}{i!}x^i \\ &= \sum_{i=1}^{n} (-1)^{(i-1)}\frac{x^i}{i} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \pm \frac{x^n}{n}. \end{aligned}$$

(b) Find the Maclaurin series expansion of f.

The Maclauin series expansion is just the Taylor series expansion about $x = 0$. Since we've already found the Taylor polynomials about $x = 0$, we can use those formulas.

$$\text{Maclaurin Series } = \sum_{i=1}^{\infty} (-1)^{(i-1)} \frac{x^i}{i}$$