Math 172 Solutions Exam 3 April 20, 2006

1. (15)

a. Define what $\lim_{n \to \infty} a_n = 4$ means,

This means that for any $\epsilon > 0$, there is an N such that if $n > N$, then

$$|a_n - 4| < \epsilon .$$

b. determine $\lim_{n \to \infty} \frac{n+1}{3n+2}$,

$$\begin{aligned} \lim_{n \to \infty} \frac{n+1}{3n+2} &= \lim_{n \to \infty} \frac{1 + 1/n}{3 + 2/n} \\ &= \frac{1}{3} . \end{aligned}$$

c. use the definition of the limit of a sequence to show that your answer to part b. is correct.

Let $\epsilon > 0$. Set $N = \frac{1/\epsilon - 6}{9}$ (Note, this is equivalent to $\frac{1}{3(3n+2)} < \epsilon$), then for $n > N$ we have

$$\begin{aligned} \left| \frac{n+1}{3n+2} - \frac{1}{3} \right| &= \left| \frac{3n+3-3n-2}{3(3n+2)} \right| \\ &= \left| \frac{1}{3(3n+2)} \right| < \epsilon . \end{aligned}$$

2. (10) Let $a_1 = 1$ and $a_{n+1} = \frac{a_n + 1}{n}$ for $n = 1, 2, \cdots$.

a. Show that $0 \le a_n \le 3$ for $n = 1, 2, \cdots$. Hint: use induction.

Since $0 \le 1 \le 3$, we see that $0 \le a_1 \le 3$. The value of a_2 is 2, which also satisfies the inequalities. Now assume that the inequalities are true for a_n with $n \ge 2$. Then we have

$$\begin{aligned} &0 \le a_n \le 3 \\ &0 \le \frac{a_n}{n} \le \frac{3}{n} \\ &\frac{1}{n} \le \frac{a_n}{n} + \frac{1}{n} \le \frac{3}{n} + \frac{1}{n} \\ &\frac{1}{n} \le \frac{a_n + 1}{n} \le \frac{4}{n} \\ &\frac{1}{n} \le a_{n+1} \le \frac{4}{n} \le \frac{4}{2} < 3 . \end{aligned}$$

Thus, the inequalities are true for all n.

b. determine the limit of a_n as $n \to \infty$.

The squeeze theorem tells us that the limiting value of a_n must be zero. For all n we have $0 \le a_n \le 3$. Thus,

$$0 \le a_{n+1} \le \frac{4}{n},$$

and the limit as $n \to \infty$ of $4/n$ is zero.

3. (15) Find the following limits

a. $\lim_{n\to\infty}\left[2+\left(\frac{-1}{n}\right)^n\right]$, b. $\lim_{n\to\infty}\frac{n^3}{n!}$, c. $\lim_{n\to\infty}[\ln(2n-1)-\ln n]$

a. $\lim_{n\to\infty}\left[2+\left(\frac{-1}{n}\right)^n\right] = 2+\lim_{n\to\infty}\left[\left(\frac{-1}{n}\right)^n\right] = 2.$

b. The terms $\frac{n^3}{n!}$, for $n > 3$, satisfy

$$\begin{aligned} 0 &\le \frac{n\bullet n\bullet n}{n(n-1)(n-2)}\bullet\frac{1}{(n-3)!} \\ &\le \frac{n\bullet n\bullet n}{n(n-1)(n-2)}\frac{1}{n-3} \\ &= \frac{1}{(1-1/n)(1-2/n)}\frac{1}{n-3}\ . \end{aligned}$$

Since this last sequence converges to zero as $n\to\infty$, the squeeze theorem tells us that $\frac{n^3}{n!}$ must also converge to zero as $n\to\infty$.

c. $\lim_{n\to\infty}[\ln(2n-1)-\ln n] = \lim_{n\to\infty}\left[\ln\left(\frac{2n-1}{n}\right)\right] = \lim_{n\to\infty}\left[\ln\left(2-\frac{1}{n}\right)\right] = \ln 2.$

4. (30) Determine whether each of the following series diverges, converges conditionally, or converges absolutely:

a. $\sum_{n=1}^{\infty}\frac{\sin n}{n\sqrt{n}}$, b. $\sum_{n=3}^{\infty}\frac{1}{n^2-4}$, c. $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+2}$, d. $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$

a. Since $\left|\frac{\sin n}{n\sqrt{n}}\right| \le \frac{1}{n^{3/2}}$, and the series $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges the original series converges absolutely.

b. The terms $\frac{1}{n^2-4}$ look like $\frac{1}{n^2}$ for large n. In fact $\frac{1}{n^2-4} \le \frac{2}{n^2}$ for $n \le 3$. Thus the original series also converges absolutely.

c. The series $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+2}$ does not converge absolutely as the absolute value of the summands look like $\frac{1}{n}$ for large n. That is for $n > 1$ we have

$$\frac{1/2}{n} < \frac{n}{n^2+2}\ .$$

Since the series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges so must the series $\sum_{n=1}^{\infty}\frac{n}{n^2+2}$. However, the terms $\frac{n}{n^2+2}$ are monotonically decreasing, for $n \ge 1$, and the limiting value of this sequence is 0. Thus, the alternating series test says that the series converges. Hence this series is conditionally convergent.

d. The series $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges by the integral test. Note $f(x) = \frac{1}{x\ln x}$ is a decreasing function of x with limit as $x\to\infty$ equal to 0.

$$\int_2^{\infty}\frac{dx}{x\ln x} = \int_{\ln 2}^{\infty}\frac{du}{u} = \ln u\Big|_{\ln 2}^{\infty} = \infty\ .$$

5. (10) Find the radius of convergence and the interval of convergence for each of the following power series:

a. $\sum_{n=0}^{\infty} \frac{nx^n}{10^n}$, b. $\sum_{n=0}^{\infty} n!(x-2)^n$

a. $\lim_{n\to\infty} \frac{\frac{(n+1)|x|^{n+1}}{10^{n+1}}}{\frac{n|x|^n}{10^n}} = \lim_{n\to\infty} \frac{(n+1)|x|}{10n} = \frac{|x|}{10}$ Since we need $\frac{|x|}{10} < 1$, the radius of convergence is 10. Moreover the series with $x = \pm 10$ diverges, as the summands do not converge to 0 as $n \to \infty$. So the interval of convergence is $\left(-10, 10\right)$.

b. $\lim_{n\to\infty} \frac{(n+1)!|x-2|^{n+1}}{n!|x-2|^n} = \lim_{n\to\infty}(n+1)|x-2| = \begin{cases} 0, & x = 2 \\ \infty, & x \neq 2 \end{cases}$. Thus, the radius of convergence is 0, and the interval of convergence is just the single point $x = 2$.

6. (10) Find the Maclaurin series expansion for $f(x) = \frac{x}{x-3}$, and determine the interval of convergence for this power series.

$$\begin{aligned} \frac{x}{x-3} &= -x\frac{1}{3-x} \\ &= -\frac{x}{3}\frac{1}{1-(x/3)} \\ &= -\frac{x}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n \\ &= -\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^{n+1} = -\sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^n \end{aligned}$$

The radius of convergence is 3. and the series does not converge at $x = \pm 3$. The summands are 1 at $x = 3$ and $(-1)^n$ at $x = -3$, and neither of these sequences converge to 0 as $n \to \infty$.

7. (10) The integral test is an *if and only if* type of theorem. State this theorem, and prove either the *if*, or the *only if* part of it.

Suppose $f(x)$ is monotonically decreasing to 0 as $x \to \infty$. Then the series

$$\sum_{n=1}^{\infty} f(n)$$

converges if and only if the integral $\int_1^{\infty} f(x)\,dx$ converges. For a proof of this theorem see your text.