All work and answers must be in your bluebook.

1. (15) Define the following:

a. The definite integral of f over the interval $[a,b]$.

The definite integral of f over the interval $\left[a, b\right]$ is defined as follows. Let P be a partition of $\left[a, b\right]$ with partition points x_i, $0 \le i \le n$. Let ξ_i be any point in the i^{th} subinterval. That is, $x_{i-1} \le \xi_i \le x_i$. Then a Riemann sum of f over the interval $\left[a, b\right]$ with respect to this partition is $\sum_{i=1}^{n} f(x_{\xi_i})(x_i - x_{i-1})$. The definite integral of f is the limit of a sequence of such Riemann sums where the norms of the partitions approach 0.

b. Define what it means to say that $\lim_{x \to 2} f(x) = 5$.

For any $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - 2|$, then $|f(x) - 5| < \epsilon$.

c. State the fundamental theorem of calculus.

There are two versions, and they are:

1. Let $f(x)$ be continuous on the interval $\left[a, b\right]$. Then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

2. Let $F(x)$ be any antiderivative of f, where f is continuous on the interval $\left[a, b\right]$.

Then

$$\int_a^b f(x)\,dx = F(b) - F(a)\ .$$

2. (10) Use the definition of the definite integral to calculate $\int_1^3 (x^2 - x)\,dx$.

Set $x_i = 1 + \frac{2i}{n}$, for $0 \le i \le n$. Then we have

$$\int_1^3 (x^2 - x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} (x_i^2 - x_i)\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(1 + \frac{2i}{n}\right)^2 - \left(1 + \frac{2i}{n}\right)\right)\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(1 + \frac{2i}{n}\right)\right)\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n} + \frac{4i^2}{n^2}\right)\frac{2}{n} = \lim_{n\to\infty}\left[\left(\frac{4}{n^2}\,\frac{n(n+1)}{2}\right) + \left(\frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6}\right)\right]$$

$$= 2 + \frac{8}{3} = \frac{14}{3}\ .$$

3. (10) Evaluate the following limit

$$\lim_{n \to \infty} \frac{3}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{1 + \frac{3i}{n}}} .$$

This is a limit of Riemann sums. Set $x_i = 1 + \frac{3i}{n}$, then $\Delta x_i = \frac{3}{n}$, the interval of integration is $\left[1, 4\right]$, and $f(x) = 1/\sqrt{x}$. Thus, we have

$$\lim_{n \to \infty} \frac{3}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{1 + \frac{3i}{n}}} = \int_1^4 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\,\Big|_1^4 = 2(2-1) = 2 .$$

4. (15) A cylindrical tank 10 meters tall and radius 5 meters is half full of water. There is a drain hole 1 meter below the top of the tank. How much work is required to pump all of the water up to the level of the drain hole and thus empty the tank?
($\rho\left(\text{water}\right) = 1000$ kg per cubic meter, acceleration due to gravity is 9.8 meters per second squared)

Let ΔW denote the work needed to lift a thin slice of water up to the drain hole. Set the origin at the top of the tank and let x denote the distance from the slice of water to the tank's top. The weight of this slice of water is

$$(9.8)25\pi\rho\Delta x \quad \text{(gravitational attraction times mass)}$$

The work needed to lift this slice to the drain hole is $(x-1)$ times the weight of the slice. The total work is the sum of all of the ΔW's and then take the limit as the Δx's tend to zero. This leads to the following definite integral:

$$\begin{aligned} \text{Work} &= \int_5^{10} (x-1)(9.8)25\pi\rho \, dx \\ &= (9.8)25000\pi \int_5^{10} (x-1)\, dx \\ &= (9.8)25000\pi \frac{65}{2} \\ &\approx 2.5 \times 10^7 \text{ Joules} \end{aligned}$$

5. (10) The function $T(x)$ is defined as

$$T(x) = \int_0^x \frac{1-\cos t}{t^2}\, dt ,$$

where $\frac{1-\cos t}{t^2}$ is defined to be 1/2 when $t = 0$.

a. What is the value of $T(0)$?

$$T(0) = \int_0^0 \frac{1-\cos t}{t^2}\, dt = 0$$

b. What is the derivative of $T(x)$?

$$\frac{d}{dx}\left(\int_0^x \frac{1-\cos t}{t^2}\, dt\right) = \begin{cases} \frac{1-\cos x}{x^2}, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases} .$$

Note: setting the value of the integrand equal to 1/2 at $x = 0$ makes the integrand continuous everywhere, which means we can apply the Fundamental Theorem of Calculus.

6. (20) Evaluate the following expressions:

a. $\int_0^{\pi} t\cos(t^2)\, dt,$

$$\int_0^{\pi} t\cos(t^2)\, dt = \left.\frac{\sin(t^2)}{2}\right|_0^{\pi} = \frac{\sin(\pi^2)}{2}.$$

b. $\int \frac{3t}{1+t^2}\, dt$

Set $u = 1 + t^2$, $du = 2t\, dt$

$$\int \frac{3t}{1+t^2}\, dt = \int \frac{(3/2)du}{u} = \frac{3}{2}\ln u + c = \frac{3}{2}\ln(1+x^2) + c.$$

c. $\int_0^1 (6x^3 - 4)(3x^4 - 8x + 6)\, dx$

Set $u = 3x^4 - 8x + 6$, then $du = (12x^3 - 8)\, dx = 2(6x^3 - 4)\, dx$.

$$\int_0^1 (6x^3 - 4)(3x^4 - 8x + 6)\, dx = \int_6^1 \frac{u}{2}\, du = \left.\frac{u^2}{4}\right|_6^1$$

$$= \frac{1}{4} - \frac{36}{4} = -\frac{35}{4}$$

d. $\int_0^1 2^x\, dx$

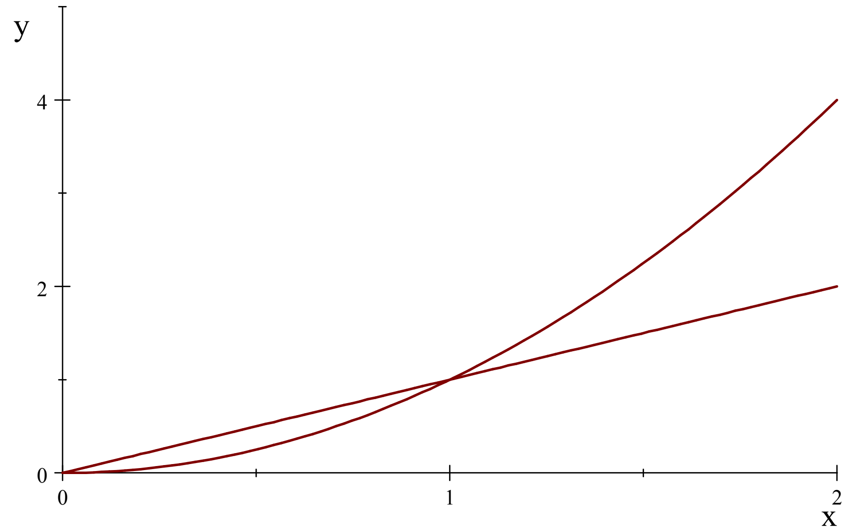
$$\int_0^1 2^x\, dx = \int_0^1 e^{x\ln 2}\, dx = \left.\frac{1}{\ln 2}e^{x\ln 2}\right|_0^1 = \frac{1}{\ln 2}(e^{\ln 2} - 1)$$

$$= \frac{1}{\ln 2}(2-1) = \frac{1}{\ln 2}\ .$$

7. (20) Let R be the region bounded by the curves $y = x^2$, $y = x$, and the line $x = 2$.

a. Find the area of the region R.

A sketch of the region R is helpful.



The area of this region is given by

$$\begin{aligned} \text{Area} &= \int_0^2 |x^2 - x|\, dx = \int_0^1 (x - x^2)\, dx + \int_1^2 (x^2 - x)\, dx \\ &= \frac{1}{6} + \frac{5}{6} \\ &= 1\,. \end{aligned}$$

b. Find the volume of the solid obtained by rotating R about the y-axis.

Easiest method is to use shells.

$$\begin{aligned} \text{Volume} &= \int_0^2 2\pi x\, |x^2 - x|\, dx = \int_0^1 2\pi x(x - x^2)\, dx + \int_1^2 2\pi x(x^2 - x)\, dx \\ &= \frac{1}{6}\pi + \frac{17}{6}\pi \\ &= 3\pi\,. \end{aligned}$$