Math 172 Solutions Exam 2 March 9, 2006

1. (15) State the mean value theorem, and use it to justify the formula below for the arc-length of a curve $\Gamma(t) = (x(t), y(t))$, $a \le t \le b$

$$\text{arc-length of curve } = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\ .$$

The mean value theorem says: let $f(x)$ be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is a point ξ in the open interval (a, b) such that

$$f(b) - f(a) = f'(\xi)(b-a)\ .$$

The formula for the arc-length of a curve is derived as follows. Let $\{t_i\}_{i=0}^n$ be a partition of $[a, b]$. Then we have

$$\begin{aligned}
\text{arc-length} &= \lim_{n \to \infty} \sum_{i=1}^{n} \|\Gamma(t_i) - \Gamma(t_{i-1})\| \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} \|(x(t_i) - x(t_{i-1}), y(t_i) - y(t_{i-1}))\| \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} \|(x'(\xi_i)(t_i - t_{i-1}), y'(\eta_i)(t_i - t_{i-1}))\|, \text{ by the mean value theorem} \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} \left([x'(\xi_i)]^2 (t_i - t_{i-1})^2 + [y'(\eta_i)]^2 (t_i - t_{i-1})^2\right)^{1/2} \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} \left([x'(\xi_i)]^2 + [y'(\eta_i)]^2\right)^{1/2} (t_i - t_{i-1}), \text{ this is a very hard step to prove} \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} \left([x'(\xi_i)]^2 + [y'(\xi_i)]^2\right)^{1/2} (t_i - t_{i-1}) \\
&= \int_a^b \left((x'(t))^2 + (y'(t))^2\right)^{1/2} dt\ .
\end{aligned}$$

The line where η_i is replaced by ξ_i is difficult to prove and usually assumes that the derivatives $x'(t)$ and $y'(t)$ are continuous functions.

2. (30) Evaluate the following integrals:

   (a) $\int x \sin x \ dx$

   Using integration by parts with $y = x$ and $dv = \sin x \ dx$ we have,

$$\begin{aligned} \int x \sin x \ dx &= -x \cos x + \int \cos x \ dx \\ &= -x \cos x + \sin x + c \ . \end{aligned}$$

   (b) $\int \frac{x^2}{\sqrt{a^2 - x^2}} \ dx$

   Make the substitution $x = a \sin \theta$, then $dx = a \cos \theta \ d\theta$, and

$$\begin{aligned} \int \frac{x^2}{\sqrt{a^2 - x^2}} \ dx &= \int \frac{a^2 \sin^2 \theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} (a \cos \theta) \ \ d\theta \\ &= a^2 \int \sin^2 \theta \ d\theta = a^2 \int \frac{1 - \cos 2\theta}{2} d\theta \\ &= a^2 \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right) + c \\ &= a^2 \left(\frac{\theta}{2} - \frac{\sin \theta \cos \theta}{2} \right) + c \\ &= \frac{a^2}{2} \left(\sin^{-1} \left(\frac{x}{a} \right) - \sin \left(\sin^{-1} \left(\frac{x}{a} \right) \right) \cos \left(\sin^{-1} \left(\frac{x}{a} \right) \right) \right) + c \\ &= \frac{a^2}{2} \left(\sin^{-1} \left(\frac{x}{a} \right) - (x/a) \frac{\sqrt{a^2 - x^2}}{a} \right) + c \\ &= \frac{a^2}{2} \left(\sin^{-1} \left(\frac{x}{a} \right) - \frac{x\sqrt{a^2 - x^2}}{a^2} \right) + c \end{aligned}$$

   (c) $\int \frac{dx}{x^2 (1 + x)}$

   Using partial fractions we have

$$\begin{aligned} \int \frac{dx}{x^2 (1 + x)} &= \int \left[\frac{1}{1 + x} - \frac{1}{x} + \frac{1}{x^2} \right] dx \\ &= \ln |1 + x| - \ln |x| - \frac{1}{x} + c \\ &= \ln \left| \frac{1 + x}{x} \right| - \frac{1}{x} + c \end{aligned}$$

3. (15) Suppose we have a "non-linear" spring such that the force, $F(x)$, required to hold the spring a length of x cm beyond its equilibrium position is given by

$$F(x) = 3 \ln (1 + x) \ .$$

How much work is done by stretching the spring from a length of 2cm to 5cm ?

$$\begin{aligned}
\text{Work} &= \int_2^5 3\ln(1+x)\ dx \\
&= 3\left[(1+x)\ln(1+x) - x\right]\Big|_2^5 \\
&= 18\ln 6 - 9\ln 3 - 9 \\
&= 9\ln\frac{36}{3} - 9 = 9\ln 12 - 9\ .
\end{aligned}$$

4. (10) Angela wishes to compute the value of $\int_0^2 e^{-x^2}\ dx$, and decides to do so by using the midpoint approximation, MP, with $n = 4$. We know that the error is bounded by

$$\left|\int_a^b f(x)\ dx - \mathrm{MP}\right| \leq \frac{K(b-a)^3}{24n^2},$$

where n is the number of subintervals, and K is such that $\left|f^{(2)}(x)\right| \leq K$ for all x in the interval $[a, b]$.

(a) Write out the value of MP for $n = 4$. You do not have to evaluate any functions, but do show where functions are being evaluated.

In the midpoint formula the integrand is evaluated at the midpoint of each subinterval. For the interval [0, 2] with $n = 4$, these points are 1/4, 3/4, 5/4, and 7/4

$$\mathrm{MP} = \frac{2}{4}\left[e^{-(1/16)} + e^{-(9/16)} + e^{-(25/16)} + e^{-(49/16)}\right]$$

(b) If Angela wanted the error to be less than 10^{-6}, what is the smallest value of n she should pick?

To use the error estimate we need to find an upper bound on the second derivative of e^{-x^2} for x in [0, 2].

$$\begin{aligned}
\frac{d^2}{dx^2}\left(e^{-x^2}\right) &= 4x^2e^{-x^2} - 2e^{-x^2} \\
&= 2e^{-x^2}\left(2x^2 - 1\right) \\
&\leq 2(1)(8-1) \\
&= 14\ .
\end{aligned}$$

The error in using the midpoint formula is bounded by

$$\text{Error} \leq \frac{K(b-a)^3}{24n^2} = \frac{14(2)^3}{24n^2} = \frac{14}{3n^2} \leq 10^{-6}\ ,$$

and this implies that n must satisfy the inequalities

$$\begin{aligned}
n^2 &\geq \frac{14}{3}10^6 \\
n &\geq \sqrt{\frac{14}{3}}10^3\ .
\end{aligned}$$

5. (20) The size of a gaggle of geese satisfies an exponential growth law. That is, if $P(t)$ is the number of geese in the gaggle at time t in years, then we have

$$\frac{dP}{dt} = \alpha P \ ,$$

for some constant α.

(a) If $\alpha = 5$, and $P(0) = 10$, how many geese will we have in the gaggle after 10 years?

The solution to $P' = 5P$ is given by $P(t) = ce^{5t}$. Since $P(0) = 10$ we have $P(t) = 10e^{5t}$. So after 10 years the gaggle will have

$$P(10) = 10e^{50} \approx 5.184 \times 10^{22} \ .$$

Anyone hungry?

(b) What must α equal in order that the population of the gaggle double every 4 years?

The solution to the differential equation with generic α is given by $P(t) = ce^{\alpha t}$. Since we want $P(4) = 2P(0)$ this means we must have

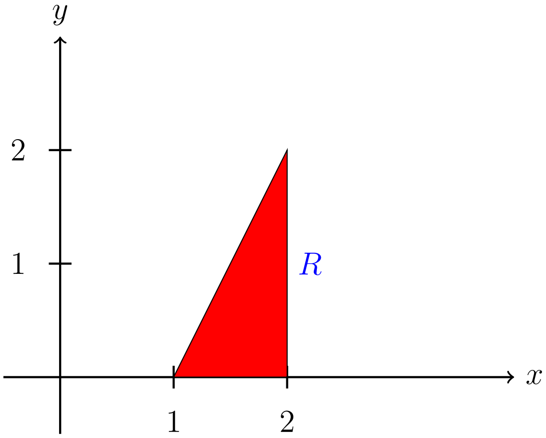
$$\begin{aligned} ce^{4\alpha} &= 2c \\ 4\alpha &= \ln 2 \\ \alpha &= \frac{\ln 2}{4} \ . \end{aligned}$$

6. (10) Let R be the region in R^2 bounded by the curves $y = 2(x-1)$, $y = 0$, $x = 1$, and $x = 2$. Let Ω_x and Ω_y denote the solids of revolution obtained by rotating R about the x-axis and y-axis respectively.

   (a) What is the surface area of Ω_x?

   (b) What is the surface area of Ω_y?

Before calculating the surface areas plot the region R, which is shown below.



a. The surface of Ω_x consists of two parts: that generated by the slanted line of R and that generated by the vertical line. Thus, the surface area of Ω_x equals

$$\begin{aligned}\text{surface area} &= \text{area due to slanted line plus vertical line}\\ &= 2\pi\int_1^2 2(x-1)\sqrt{1+4}\,dx + \pi\left(2^2\right)\\ &= 2\pi\sqrt{5}+4\pi = 2\pi\left(2+\sqrt{5}\right).\end{aligned}$$

b. The surface of Ω_y consists of three parts: that generated by the slanted line of R, that by the vertical line, and that by the horizontal line. Thus, the surface area of Ω_y equals

$$\begin{aligned}\text{surface area} &= \text{area by slanted line plus vertical line plus horizontal line}\\ &= 2\pi\int_0^2 (y/2+1)\sqrt{1+1/4}\,dy + (2\pi)\times 2\times 2 + (4\pi-\pi)\\ &= 2\pi\left(\frac{3}{2}\sqrt{5}\right)+8\pi+3\pi\\ &= 3\sqrt{5}\pi\ +11\pi = \left(3\sqrt{5}+11\right)\pi.\end{aligned}$$