All work and answers must be in your bluebook

1. (15) Evaluate the following limits:

a.

$$\lim_{n\to\infty} \frac{n+2}{2n+1} = \lim_{n\to\infty} \frac{1+2/n}{2+1/n} = \frac{1}{2}\,.$$

b.

$$\begin{aligned}
\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n &= \lim_{n\to\infty} e^{n\ln(1-1/n)} = e^{\lim_{n\to\infty} n\ln(1-1/n)} \\
\lim_{n\to\infty} n\ln(1-1/n) &= \lim_{n\to\infty} \frac{\ln(1-1/n)}{1/n} = -1\,.\ \text{Thus,} \\
\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n &= e^{-1}\,.
\end{aligned}$$

c. $\displaystyle\lim_{n\to\infty} \frac{(-3)^n}{n!}$

We have, for $n > 3$ the following inequality (actually the inequality holds for all positive integers n):

$$0 \le \left|\frac{(-3)^n}{n!}\right| = \frac{3\cdot 3\cdot 3\cdot 3\cdots 3}{1\cdot 2\cdot 3\cdot 4\cdots n} \le \left(\frac{9}{2}\right)\left(\frac{3}{n}\right)\,.$$

Thus, by the squeeze theorem, the limiting value of this sequence must be zero.

2. (10) Using the definition of limit, prove that your answer to question 1a. is correct.

Let $\epsilon > 0$, set $N = \frac{3}{\epsilon}$. Note that $n > 3/\epsilon$ if and only if $3/n < \epsilon$. Now, if $n > N$ we have

$$\begin{aligned}
\left|\frac{n+2}{2n+1} - \frac{1}{2}\right| &= \left|\frac{(2n+4)-(2n+1)}{2(2n+1)}\right| = \frac{3}{2(2n+1)} \\
&< \frac{3}{n} < \epsilon
\end{aligned}$$

3. (20)

a. State the monotone convergence theorem.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that the sequence is monotone and bounded. Then the sequence has a limit as n goes to infinity. That is, we assume either $a_n \leq a_{n+1}$ for all n or $a_n \geq a_{n+1}$ for all n, and there is an M such that $|a_n| \leq M$ for all n. Under these conditions we know $\lim\limits_{n\to\infty} a_n$ exists.

b. Let $0 \leq a_n \leq b_n$. Show that if the series $\sum\limits_{n=0}^{\infty} b_n$ converges then so does the series $\sum\limits_{n=0}^{\infty} a_n$.

An infinite series converges if its sequence of partial sums converges. Let $S_n = \sum\limits_{j=0}^{n} a_j$. Since the terms $a_n \geq 0$ the sequence of partial sums is an increasing sequence. Moreover we have

$$S_n = \sum_{j=0}^{n} a_j \leq \sum_{j=0}^{n} b_j \leq \sum_{j=0}^{\infty} b_j = M \text{ a finite number.}$$

Thus, the sequence S_n is bounded and monotone, and by the monotone convergence theorem it must converge.

4. (35) Determine whether the following series diverge, converge conditionally, or converge absolutely.

a. $\sum\limits_{n=1}^{\infty} \frac{\sqrt{n}}{2n+5}$, b. $\sum\limits_{n=2}^{\infty} \frac{3}{n(n-1)}$, c. $\sum\limits_{n=1}^{\infty} (-1)^n \ln\left(\frac{n}{2n+5}\right)$, d. $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

a. This series diverges. The n^{th} term $\frac{\sqrt{n}}{2n+5}$ satisfies the inequality

$$\frac{\sqrt{n}}{2n+5} \geq \frac{\sqrt{n}}{7n} = \frac{1}{7\sqrt{n}},$$

for any postive integer n. Since the series $\sum \frac{1}{\sqrt{n}}$ diverges (it's a p-series with $p \leq 1$), we know the series $\frac{1}{7\sqrt{n}}$ diverges, and the comparison test tells us the given series diverges.

b. The series $\displaystyle\sum_{n=2}^{\infty} \frac{3}{n(n-1)}$ converges and since its terms are positive, it converges absolutely. The n^{th} term satisfies the inequality

$$\frac{3}{n(n-1)} \leq \frac{3}{(n-1)^2}\,.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the comparison test tells us the given series converges.

c. The series $\sum_{n=1}^{\infty} (-1)^n \ln\left(\frac{n}{2n+5}\right)$ diverges. If the series converged then the limit as $n \to \infty$ of the terms must be zero, but

$$\lim_{n\to\infty} \ln\left(\frac{n}{2n+5}\right) = \ln(\frac{1}{2}) \neq 0\,.$$

d. The series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges conditionally. The alternating series test gives convergence and the integral test shows that the series does not converge absolutely.

5. (20) Find the radius and interval of convergence for each of the following series;

a. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n+1}$,

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+2}}{\frac{x^n}{n+1}} \right| = \lim_{n\to\infty} |x| \left(\frac{n+1}{n+2}\right) = |x|\,.$$

Since we need the limit to be less than 1, we need $|x| < 1$. So the radius of convergence is 1. Checking what happens at the endpoints ± 1 we see that the series diverges at 1, and converges at -1. Thus, the interval of convergence is $[-1, 1)$.

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-4)^n}{n^2 5^n}$

$$\lim_{n\to\infty} \left| \frac{\frac{(x-4)^{n+1}}{(n+1)^2 5^{n+1}}}{\frac{(x-4)^n}{n^2 5^n}} \right| = \lim_{n\to\infty} |x-4| \left(\frac{n}{n+1}\right)^2 \frac{1}{5} = \frac{|x-4|}{5}\,.$$

Thus, we need to have $|x-4| < 5$, which means the radius of convergence is 5. At the endpoints -1 and 9 the series converges absolutely. Thus, the interval of convergence is $[-1, 9]$.