All work and answers must be in your bluebook

1. (20) The definite integral of f from a to b, $\int_a^b f(x)\,dx$, is defined as a limit of a sequence of Riemann sums.

   a. Define what a Reimann sum of f over the interval $[a, b]$ is.

      Let $\{x_i\}_{i=0}^n$ be a partition of $[a, b]$. That is, $a = x_0 < x_1 < \cdots < x_n = b$. Let t_i be any point in the i^{th} subinterval. That is, $x_{i-1} \leq t_i \leq x_i$ for each $i = 1, 2, \cdots, n$. Then a Reimann sum is

$$\sum_{i=1}^{n} f(t_i)\,(x_i - x_{i-1})\ .$$

   b. Using the definition of a definite integral show that if $g(x) \leq f(x)$ for $a \leq x \leq b$, then $\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx$.

      Consider two Riemann sums, one for f and one for g, both of which use the same partition and the same points t_i. We then have

$$\sum_{i=1}^{n} g(t_i)\,(x_i - x_{i-1}) \leq \sum_{i=1}^{n} f(t_i)\,(x_i - x_{i-1})\ .$$

      Suppose P_n is a sequence of partitions with the norms of the partitions going to zero as $n \to \infty$. We have the above inequality for each of the partitions, and as $n \to \infty$ the Riemann sums converge to the integrals of g and f respectively. Thus, giving us the desired inequality.

2. (30) Calculate the following integrals:

   a. $\int_0^3 \frac{1}{9+x^2}\,dx$.

      First make the substitution $x = 3u$.

$$\int_0^3 \frac{1}{9+x^2} = \int_0^1 \frac{3\,du}{9+9u^2} = \frac{1}{3}\int_0^1 \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1} u\Big|_0^1 = \frac{\pi}{12}$$

b. $\displaystyle\int \frac{x\,dx}{(1-x)(1+x^2)}$. Find the partial fraction decomposition of the integrand. That is, find constants a, b, c such that

$$\frac{x}{(1-x)(1+x^2)} = \frac{a}{1-x} + \frac{bx+c}{1+x^2}.$$

This leads to the equations:

$$\begin{aligned}
x &= a(1+x^2) + (1-x)(bx+c) = (a-b)x^2 + (b-c)x + (a+c) \implies \\
0 &= a-b,\ 1 = b-c,\ 0 = a+c \implies \\
a &= \frac{1}{2},\ b = \frac{1}{2},\ c = -\frac{1}{2}
\end{aligned}$$

Hence,

$$\begin{aligned}
\int \frac{x\,dx}{(1-x)(1+x^2)} &= \frac{1}{2}\left(\int \frac{dx}{1-x} + \int \frac{x-1}{1+x^2}\,dx\right) \\
&= \frac{1}{2}\left(\int \frac{dx}{1-x} + \int \frac{x}{1+x^2}\,dx - \int \frac{1}{1+x^2}\,dx\right) \\
&= -\frac{1}{2}\ln|1-x| + \frac{1}{4}\ln(1+x^2) - \frac{1}{2}\tan^{-1}x + c\,.
\end{aligned}$$

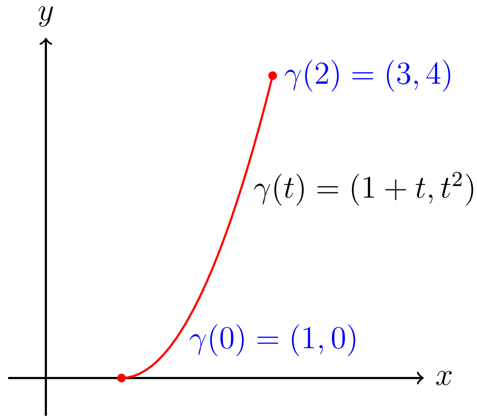
c. $\displaystyle\int_0^{\ln 2} xe^{2x}\,dx$

This is handled using integration by parts. Set $u = x$ and $dv = e^{2x}dx$. Then we have

$$\begin{aligned}
\int_0^{\ln 2} xe^{2x}\,dx &= \frac{x}{2}e^{2x}\Big|_0^{\ln 2} - \int_0^{\ln 2} \frac{e^{2x}}{2}\,dx \\
&= 2\ln 2 - \frac{e^{2x}}{4}\Big|_0^{\ln 2} = 2\ln 2 - \left(\frac{4}{4} - \frac{1}{4}\right) \\
&= 2\ln 2 - \frac{3}{4}\,.
\end{aligned}$$

3. (30) Let $\gamma(t) = (1+t, t^2)$, for $0 \le t \le 2$. In parts b. through d. express your answer as an integral. You do not have to evaluate it.

a. Graph the curve γ,



b. What is the length of the curve γ,

$$\text{Length of } \Gamma = \int_0^2 \left(1 + (2t)^2\right)^{1/2}\, dt$$

c. What is the surface area of the surface generated by rotating the curve γ about the y-axis.

$$\text{Surface area } = 2\pi \int_0^2 (1+t)\left(1 + (2t)^2\right)^{1/2}\, dt$$

d. What is the volume of the region obtained by rotating the curve γ about the x-axis.

$$\text{Volume of region } = \int_0^2 \pi (t^2)^2\, dt$$

4. (20) Let $f(x) = \sqrt{1+x}$.

   a. What is the third order (degree) Taylor polynomial of f about $x = 0$.

   We first need to calculate the first 3 derivatives of f and evaluate them at $x = 0$.

$$\begin{aligned} f(x) &= (1+x)^{1/2}, f'(x) = \frac{1}{2}(1+x)^{-1/2}, f''(x) = -\frac{1}{4}(1+x)^{-3/2}, f^{(3)}(x) = \frac{3}{8}(1+x)^{-5/2} \\ f(0) &= 1,\ f'(0) = \frac{1}{2},\ f''(0) = -\frac{1}{4},\ f^{(3)}(0) = \frac{3}{8} \end{aligned}$$

$$\begin{aligned} T^3_{f,0}(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 \\ &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}. \end{aligned}$$

   b. Use your answer to part a. to approximate $\sqrt{1.1}$.

$$\sqrt{1.1} = f(0.1) \approx T^3_{f,0}(0.1) = 1 + \frac{0.1}{2} - \frac{(0.1)^2}{8} + \frac{(0.1)^3}{16} \approx 1.048812500$$

   As an aside, the square root of 1.1, correct to a large number of places, is

$$1.0488088481701515469914\,54.$$

5. (25) State the integral test, which is used to determine the convergence or divergence of certain types of infinite series. Then use it to decide if the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ converges or diverges.

Suppose the terms a_n of the series $\displaystyle\sum_{n=1}^{\infty} a_n$ are given by a function $f(x)$. That is $a_n = f(n)$. Suppose that $f(x) > 0$ and is a decreasing function of x on the interval $[1, \infty)$. Then the integral of f converges if and only if the series converges. That is,

$$\int_1^{\infty} f(x)\,dx < \infty \text{ if and only if } \sum_{n=1}^{\infty} a_n < \infty$$

To see if the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ converges, set $f(x) = \frac{1}{x(\ln x)^3}$. f is positive and decreasing on the interval $(1, \infty)$, and we have

$$\int_2^{\infty} \frac{1}{x(\ln x)^3} = \int_{\ln 2}^{\infty} \frac{du}{u^3} = \left.\frac{-1/2}{u^2}\right|_{\ln 2}^{\infty} = \frac{1}{2\ln 2} < \infty$$

Since the integral is finite so is the series, i.e., the series converges.

6. (25) Find the radius and interval of convergence of the following power series:

a. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}(x-3)^n,$

Use the ratio test.

$$\lim_{n\to\infty}\left|\frac{\frac{(x-3)^{n+1}}{2^{n+1}}}{\frac{(x-3)^n}{2^n}}\right| = \frac{|x-3|}{2}$$

Since this ratio needs to be less than 1, we see that $|x-3| < 2$. So the radius of convergence is 2. The endpoints of the interval of convergence are 1 and 5. Setting x =1 and 5 in the series we get respectively

$$\sum_{n=0}^{\infty}\frac{1}{2^n}(-2)^n = \sum_{n=0}^{\infty}(-1)^n, \text{ which diverges.}$$

$$\sum_{n=0}^{\infty}\frac{1}{2^n}(2)^n = \sum_{n=0}^{\infty}1, \text{ which also diverges.}$$

Thus, the interval of convergence is $(1, 5)$.

b. $\displaystyle\sum_{n=0}^{\infty}\frac{(x+1)^n}{n}.$

The ratio test is used again.

$$\lim_{n\to\infty}\left|\frac{\frac{(x+1)^{n+1}}{n+1}}{\frac{(x+1)^n}{n}}\right| = |x+1|\lim_{n\to\infty}\frac{n}{n+1} = |x+1|$$

The radius of convergence is 1, and the endpoints of the interval of convergence are -2 and 0. Putting those values into the series we have

$$\text{Setting } x = -2 \text{ in the sum, we get } \sum_{n=0}^{\infty}\frac{(-1)^n}{n}, \text{ which converges.}$$

$$\text{Setting } x = 0 \text{ in the sum, we get } \sum_{n=0}^{\infty}\frac{1}{n}, \text{ which diverges.}$$

Thus, the interval of convergence is $[-2, 0)$.