

Suppose that  $a_n$  for  $n = 1, 2, \dots$  are positive numbers, that  $a_{n+1} \leq a_n$ , and  $\lim_{n \rightarrow \infty} a_n = 0$ , then the series

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n,$$

converges, and the error in approximating the sum  $S$  with the  $n^{\text{th}}$  partial sum is less than the absolute value of the first omitted term.

Let  $S_n$  denote the  $n^{\text{th}}$  partial sum of the infinite series. That is,  $S_n = \sum_{i=1}^n (-1)^{i-1} a_i$ .

We will show that the even indexed partial sums form an increasing sequence and the odd indexed partial sums form a decreasing sequence. That is,

$$S_{2n} \leq S_{2(n+1)} \text{ for } n = 1, 2, \dots \quad \text{and} \quad S_{2n+1} \leq S_{2(n-1)+1} \text{ for } n = 1, 2, \dots \quad (1)$$

The following lines verify these two inequalities. Note that the index  $i$  of  $a_i$  has the opposite parity of the exponent of  $(-1)$ . That is, even indexed terms are preceded by a minus sign and odd indexed terms are preceded by a plus sign.

$$\begin{aligned} S_{2n} &= \sum_{i=1}^{2n} (-1)^{i-1} a_i \leq \sum_{i=1}^{2n} (-1)^{i-1} a_i + a_{2n+1} - a_{2n+2} = S_{2n+2} = S_{2(n+1)} \\ S_{2n+1} &= \sum_{i=1}^{2n+1} (-1)^{i-1} a_i = \sum_{i=1}^{2(n-1)+3} (-1)^{i-1} a_i = S_{2(n-1)+1} - a_{2n} + a_{2n+1} \leq S_{2(n-1)+1} \end{aligned}$$

One other observation: for any  $n$  we have  $S_{2n} \leq S_{2(n-1)+1}$ . That is, the  $n^{\text{th}}$  even partial sum is less than or equal to the  $n^{\text{th}}$  odd partial sum. The following line demonstrates this:

$$S_{2(n-1)+1} = S_{2n-1} = \sum_{i=1}^{2n-1} (-1)^{i-1} a_i \geq \sum_{i=1}^{2n-1} (-1)^{i-1} a_i - a_{2n} = \sum_{i=1}^{2n} (-1)^{i-1} a_i = S_{2n}.$$

One last comment: every even indexed partial sum is less than or equal to any odd indexed partial sum. To see this suppose  $m < n$ . Then we have

$$S_{2m} \leq S_{2n} \leq S_{2(n-1)+1}.$$

A similar argument works if  $m > n$ . Thus the even indexed partial sums form an increasing bounded sequence and the odd indexed partial sums form a decreasing bounded sequence. By the monotone convergence theorem both of these sequence must have limits. Let

$S_e = \lim_{n \rightarrow \infty} S_{2n}$  and  $S_o = \lim_{n \rightarrow \infty} S_{2n+1}$ . We know that  $S_e \leq S_o$ . Moreover

$$0 = \lim_{n \rightarrow \infty} a_{2n+1} = \lim_{n \rightarrow \infty} (S_{2n+1} - S_{2n}) = S_o - S_e.$$

Since the even and odd indexed partial sums converge to the same value the original series converges.

The error estimate is somewhat easy. Let  $S$  denote the sum of the alternating series. We know the following inequality is true:

$$0 < S_2 < S < S_1 \text{ that is } 0 < a_2 - a_1 < S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n < a_1.$$

That is, the absolute value of  $S$  is less than the absolute value of the first term in the sum. So this means that

$$|S - S_n| = \left| \sum_{i=n+1}^{\infty} (-1)^{i-1} a_i \right| < |(-1)^{n+1} a_{n+1}| = a_{n+1}$$

If the alternating series starts with a minus sign, the easiest way to see this approximation is still true is to multiply the series by  $(-1)$ , and then invoke the result for the case the series starts with a plus sign.