

1. (10) Let  $A$ ,  $B$ , and  $C$  denote three sets. Prove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) .$$

Suppose  $x \in A \cap (B \cup C)$ . Then  $x \in A$  and  $x \in (B \cup C)$ . So  $x \in B$  or  $x \in C$  as well as  $x \in A$ . Thus,  $x \in A \cap B$  or  $x \in A \cap C$ , which means  $x \in (A \cap B) \cup (A \cap C)$ . Suppose next that  $x \in (A \cap B) \cup (A \cap C)$ . Then  $x \in A \cap B$  or  $x \in A \cap C$ . Thus,  $x \in A$  and  $x \in B$ , or  $x \in A$  and  $x \in C$ . Thus,  $x \in A$  and  $x \in B$  or  $x \in C$ . Hence,  $x \in B \cup C$ , which implies that  $x \in A \cap (B \cup C)$ .

2. (15) In this problem the symbol  $R$  denotes the set of real numbers. Consider the following definition: a function  $f : R \rightarrow R$  is said to be left continuous at a point  $x_0$  if for every  $\epsilon > 0$ , there is a  $\delta > 0$  such that if

$$0 < x_0 - x < \delta, \text{ then } |f(x) - f(x_0)| < \epsilon ,$$

and we say that  $f$  is right continuous at a point  $x_0$  if for every  $\epsilon > 0$ , there is a  $\delta > 0$  such that if

$$0 < x - x_0 < \delta, \text{ then } |f(x) - f(x_0)| < \epsilon .$$

- (a) What is the negation of the definition of left continuous at  $x_0$ ?

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0 \exists x \text{ such that } 0 < x_0 - x < \delta \text{ and } |f(x) - f(x_0)| \geq \epsilon$$

- (b)  $[x]$  denotes the greatest integer function. That is,

$$[x] = n,$$

where  $n$  is that unique integer such that  $n \leq x < n+1$ . Show that  $[x]$  is right continuous at  $x_0 = 3$ .

Let  $\epsilon > 0$ . Set  $\delta = 1/2$ . Then if  $0 < x - 3 < 1/2$ , we have  $3 < x < 3.5$ . Thus  $[x] = 3$ , and  $|[x] - [3]| = 0 < \epsilon$ .

- (c) Show that  $[x]$  is not left continuous at  $x_0 = 3$ .

Set  $\epsilon = 1/2$ . Let  $\delta > 0$ . Set  $\hat{x} = 3 - \delta/2$ . Then  $3 - \hat{x} = \delta/2$  and we have  $0 < 3 - \hat{x} < \delta$ .  
And

$$\begin{aligned} |[\hat{x}] - [3]| &= |[3 - \delta/2] - 3| \\ &= |2 - 3| = 1 > 1/2 = \epsilon . \end{aligned}$$

3. (30) Let  $A = R \times R$ , the Cartesian product of the real numbers with the real numbers. Define a relation  $\mathbb{R}$  on  $A$  by

$$(x, y) \mathbb{R} (a, b)$$

if  $y - b = 3(x - a)$ .

- (a) Show that this is an equivalence relation.

We need to show that the relation  $\mathbb{R}$  is reflexive, symmetric and transitive. For reflexivity, since  $y - y = 3(x - x)$  we have  $(x, y) \mathbb{R} (x, y)$ . If  $(x, y) \mathbb{R} (a, b)$ , then  $y - b = 3(x - a)$ . This implies  $b - y = 3(a - x)$ . Thus  $(a, b) \mathbb{R} (x, y)$ , and  $\mathbb{R}$  is symmetric. For transitivity suppose

$$(x, y) \mathbb{R} (a, b) \text{ and } (a, b) \mathbb{R} (u, v) .$$

Then we have

$$y - b = 3(x - a) \text{ and } b - v = 3(a - u) .$$

Adding these two equations together we have  $y - v = 3(x - u)$ . Thus

$$(x, y) \mathbb{R} (u, v) .$$

- (b) Let  $[(x, y)]$  denote the equivalence class generated by the pair  $(x, y)$ . Define "addition" of equivalence classes by

$$[(x, y)] + [(a, b)] = [(x + a, y + b)] .$$

Show that this binary operation on the set of equivalence classes is well defined.

Suppose  $(x_1, y_1) \mathbb{R} (x_2, y_2)$  and  $(a_1, b_1) \mathbb{R} (a_2, b_2)$ . Then we want to show that  $(x_1 + a_1, y_1 + b_1) \mathbb{R} (x_2 + a_2, y_2 + b_2)$ . From the given relations we have

$$y_1 - y_2 = 3(x_1 - x_2) \text{ and } b_1 - b_2 = 3(a_1 - a_2) .$$

Adding these equations together we get the equation

$$\begin{aligned} y_1 - y_2 + b_1 - b_2 &= 3(x_1 - x_2) + 3(a_1 - a_2) \\ (y_1 + b_1) - (y_2 + b_2) &= 3((x_1 + a_1) - (x_2 + a_2)) . \end{aligned}$$

This means that  $(x_1 + a_1, y_1 + b_1) \mathbb{R} (x_2 + a_2, y_2 + b_2)$ . So no matter which representative we pick from the equivalence classes  $[(x, y)]$  and  $[(a, b)]$ , the sum is the same. That is,

$$\begin{aligned} [(x_1, y_1)] + [(a_1, b_1)] &= [(x_1 + a_1, y_1 + b_1)] \\ &= [(x_2 + a_2, y_2 + b_2)] \end{aligned}$$

- (c) Find an identity element for the binary operation of  $+$  on the equivalence classes.

The identity element is the equivalence class  $[(0, 0)]$ . For

$$[(x, y)] + [(0, 0)] = [(x + 0, y + 0)] = [(x, y)] .$$

Similarly we have  $[(0, 0)] + [(x, y)] = [(x, y)]$ .

- (d) In the Cartesian plane sketch the equivalence class  $[(1, -2)]$ .

The equivalence class is the straight line with slope 3 passing through the point  $(1, -2)$ .

4. (15) Let  $m$  and  $n$  be positive integers. Suppose the base 6 representations of these integers are

$$\begin{aligned}(m)_6 &= 135 \\ (n)_6 &= 42 .\end{aligned}$$

- (a) What integer is  $m$ ? Use base 10 notation.

$$m = 1 \cdot 6^2 + 3 \cdot 6 + 5 = 59$$

- (b) What is the base 6 representation of  $m + n$ .

There are two ways to work this problem. One is to convert to base 10, add the numbers, then convert back to base 6. The other is to just add in base 6. We'll do this problem using both methods. Staying in base 6 we note first that  $5 + 2 = 1$  with a carry of 1, and  $1 + 3 + 4 = 2$  with a carry of 1 and  $1 + 1 = 2$ . Thus,

$$135 + 42 = 221 .$$

Converting to base 10 we have  $m + n = 59 + 4 \cdot 6 + 2 = 85$ , and

$$\begin{aligned}85 &= 6 \cdot 14 + 1 \\ 14 &= 6 \cdot 2 + 2 \\ 2 &= 6 \cdot 0 + 2 .\end{aligned}$$

Thus,  $(85)_6 = 221$ . As an additional check we have

$$2 \cdot 6^2 + 2 \cdot 6 + 1 = 72 + 13 = 85 .$$

5. (15) Let  $m = 155$  and  $n = 143$ .

(a) Find the greatest common divisor of  $m$  and  $n$ .

$$\begin{aligned}155 &= 143 + 12 \\143 &= 12 \cdot 11 + 11 \\12 &= 11 \cdot 1 + 1.\end{aligned}$$

Thus, the greatest common divisor of 155 and 143 is 1.

(b) Does the equation  $143x + 155y = 3$  have any integer solutions? If yes, find one set of solutions, and if no, explain why not.

Equations of this type have a solution if and only if the greatest common divisor of 143 and 155 divides 3. Since this number is 1, the equations have a solution. To find one such solution we write 1 as a linear combination of 155 and 143.

$$\begin{aligned}1 &= 12 - 11 \\&= 12 - (143 - 11 \cdot 12) = 12 \cdot 12 - 143 \\&= 12(155 - 143) - 143 \\&= 12 \cdot 155 - 13 \cdot 143.\end{aligned}$$

Now multiply this equation by 3

$$36 \cdot 155 - 39 \cdot 143 = 3.$$

Thus,  $x = -39$  and  $y = 36$  is a solution to

$$143x + 155y = 3.$$

(c) Does the equation  $140x + 155y = 3$  have any integer solutions? If yes, find one set of solutions, and if no, explain why not.

The equation  $140x + 155y = 3$  has a solution if and only if the greatest common divisor of 140 and 155 divides 3. However, since 5 divides both 140 and 155, this means that 5 must divide 3. Since it doesn't there can be no solution.

6. (15) Let  $A$  be a non empty set. For each  $n \in \mathbb{N}$ , let  $f_n : A \rightarrow A$ . Show by induction that if  $f_n$  is injective for each  $n$ , then the composition

$$F_n = f_1 \circ f_2 \circ \cdots \circ f_n,$$

is injective for each  $n$ .

Let  $S = \{n \in \mathbb{N} : F_n \text{ is injective}\}$ . Since  $F_1 = f_1$  and  $f_1$  is injective, we have  $1 \in S$ . Suppose next that  $k \in S$ . That is  $F_k = f_1 \circ \cdots \circ f_k$  is injective. Then

$$\begin{aligned}F_{k+1} &= f_1 \circ \cdots \circ f_k \circ f_{k+1} \\&= F_k \circ f_{k+1}.\end{aligned}$$

To see that  $F_{k+1}$  is injective we need only show that the composition of two injective functions is injective. To that end suppose  $f$  and  $g$  are both injective, where

$$f : A \rightarrow B \text{ and } g : B \rightarrow C.$$

Suppose  $g \circ f(a_1) = g \circ f(a_2)$ . That is,  $g(f(a_1)) = g(f(a_2))$ . Since  $g$  is injective we have  $f(a_1) = f(a_2)$ , and the injectiveness of  $f$  implies that  $a_1 = a_2$ . Thus,  $g \circ f$  is also injective. This of course now implies that  $F_{k+1}$  is injective. Thus, by induction the set  $S = \mathbb{N}$ . That is, for all positive integers, the function  $F_n = f_1 \circ \cdots \circ f_n$  is injective.