1. (15) Define the following terms:

(a) If A and B are two sets, what does $A \subseteq B$ mean?

$A \subseteq B$ means that A is a subset of B. That is, if $x \in A$, then $x \in B$.

(b) If A and B are two sets what is $A \times B$?

$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

(c) If A is a set, what is the power set of A ?

The power set of A is the collection of all subsets of A,
$P(A) = \{B : B \subseteq A\}$

2. (20) Let $P(x,\, y)$ denote an open statement. That is, it becomes a statement when x and y are assigned particular values. In the following it is understood that x and y are integers.

a) $\forall x\ \exists y$ such that $P(x,\, y)$
b) $\exists y$ such that $\forall x\ P(x,\, y)$

(a) Determine whether or not a) implies b). If it does give a proof, if it doesn't give a counter example.

(b) Determine whether or not b) implies a). If it does give a proof, if it doesn't give a counter example.

Remember x and y represent integers so any counter example must involve integers.

The open statement $\forall x\ \exists y$ such that $P(x, y)$. Means that for every x there is a y, which may vary from one x to another, such that $P(x, y)$ is a true statement. On the other hand the open statement $\exists y$ such that $\forall x\ P(x, y)$ means that there is a y, now fixed and constant, such that $P(x, y)$ is true for any x and this fixed y. Thus, it seems that a. should not imply b. but that b. should imply a.

An example to show that a. does not imply b. is the following: let $P(x, y)$ be the open statement $x + y = 3$, where x and y denote integers. Clearly a. is true; for any x set $y = 3 - x$, then for these particular x and y $P(x, 3 - x)$ is a true statement. On the other hand it is not true that there is a fixed integer y_0 such that $x + y_0 = 3$ for every integer x. Thus, a. does not imply b.

It is clear that b. implies a. Suppose b. is true. Let y_0 denote a y for which $P(x, y)$ is true for all x. To see that a. is true for each x set $y = y_0$.

3. (25) A real valued function of a real variable x is said to be continuous at a point a if

$$\begin{aligned} \forall \epsilon &> 0 \; \exists \delta > 0 \text{ such that } \forall x \\ \text{if } |x-a| &< \delta, \text{ then } |f(x) - f(a)| < \epsilon \; . \end{aligned}$$

(a) What is the negation of the above statement?

The negation is

$$\exists \epsilon > 0 \; \forall \delta > 0 \; \exists x \text{ such that } \; |x-a| < \delta \wedge |f(x) - f(a)| \geq \epsilon$$

(b) Give an explicit example of a function that is not continuous at $a = 2$. Be sure to justify your example.

An example of a function discontinuous at $x = 2$ is

$$f(x) = \begin{cases} 1, & x \leq 2 \\ 3, & 2 < x \end{cases}$$

Set $\epsilon = 1/2$. For any $\delta > 0$ set $x = 2 + \delta/2$. Then we have

$$\begin{aligned} |x-2| &= |2 + \delta/2 - 2| = \delta/2 < \delta \text{ and} \\ |f(x) - f(2)| &= |f(2 + \delta/2) - f(2)| = |3 - 1| = 2 \geq 1/2 \end{aligned}$$

4. (20) State Peano's axiom of induction as it pertains to the set $N = \{1, 2, \cdots\}$ of natural numbers. Then use this axiom to show that for any positive integer n

$$2^{n-1} \leq n! \; ,$$

where $n! = 1(2)(3)\cdots(n)$. Thus, $4! = 24$ and $5! = 5(4!) = 120$.

Let P be any subset of N for which

$$\begin{aligned} \text{a. } 1 &\in P \\ \text{b. } n &\in P \text{ implies that } n+1 \in P \end{aligned}$$

then $P = N$.

To see that $2^{n-1} \leq n!$ for every natural number n we use induction. Let P be those $n \in N$ for which $2^{n-1} \leq n!$. It is easy to see that $1 \in P$. Now suppose $n \in P$. Then we have

$$2^{(n+1)-1} = 2 \; 2^{n-1} \leq (n+1) \, n! = (n+1)!$$

Thus, $n+1 \in P$, and by induction we have $P = N$. That is,

$$2^{n-1} \leq n! \text{ for every } n \in N \; .$$

5. (20) For each $\gamma \in \Gamma$, let A_γ be a set. Determine whether the following is or is not true. If true supply a proof, and if false supply a counter example.

$$\overline{\cup_{\gamma\in\Gamma} A_\gamma} = \cap_{\gamma\in\Gamma} A_\gamma \ .$$

The two sets are not equal. An example to see this is the following:

Let $A_1 = \{1, 2\}$ and $A_2 = \{3, 4\}$, with the understanding that the universal set is the set of natural numbers $N = \{1,\ 2,\ \cdots\}$. Then

$$\begin{aligned} \overline{A_1 \cup A_2} &= \{5,\ 6,\ 7,\ \cdots\} \text{ and} \\ A_1 \cap A_2 &= \emptyset\ . \end{aligned}$$