1. (30) Let $f : N \to N$, where N is the set of natural numbers $\{1, 2, \cdots\}$. Suppose that $f(n) = n^2 + 1$.

   (a) If f injective (one-to-one)?

   Yes f is injective. Suppose $f(n) = f(m)$. This implies

$$\begin{aligned} m^2 + 1 &= n^2 + 1 \\ m^2 &= n^2 \\ m &= \pm n \, . \end{aligned}$$

   Since m and n are natural numbers m cannot be negative. Thus $m = n$.

   (b) Is f surjective (onto)?

   No f is not surjective. 1 is a natural number, and the equation $n^2 + 1 = 1$ implies that $n^2 = 0$, or $n = 0$. Since $0 \notin N$. f is not onto.

   (c) Explain what f^{-1} must mean for this particular function.

   Since f is not onto, it cannot have an inverse function. Since f^{-1} is used to represent the f's inverse function, that meaning does not apply here. So all that is left is for f^{-1} to represent the mapping from $\mathrm{P}(B)$ to $\mathrm{P}(A)$ defined as follows: let Y be any subset of B, then

$$f^{-1}(Y) = \{a \in A : f(a) \in Y\} \; .$$

2. (30) Suppose $f : A \to B$ and $g : B \to C$.

   (a) If f and g are both one-to-one, must $g \circ f$ be one-to-one?

   Yes, the composition must also be one-to-one. Suppose a_1 and a_2 are elements of A such that

$$\begin{aligned} g \circ f(a_1) &= g \circ f(a_2) \\ g(f(a_1)) &= g(f(a_2)) \end{aligned}$$

   Since g is one-to-one we have

$$\begin{aligned} f(a_1) &= f(a_2) \\ a_1 &= a_2 \, . \end{aligned}$$

   The last equality follows from the fact that f is one-to-one.

   (b) If f and g are both onto, must $g \circ f$ be onto?

   Yes, the composition must be onto. Let c be any element in C. Since g is onto there is an element $b \in B$ such that $g(b) = c$. Since f is onto there is an $a \in A$ such that $f(a) = b$. Thus, we have

$$g(f(a)) = g(b) = c.$$

   That is, $g \circ f(a) = c$.

(c) If f is onto and g is one-to-one must $g \circ f$ be either of these.

The answer is no. Let $A = \{1, 2, 3\}$, $B = \{1, 2\}$, and $C = \{1, 2, 3\}$.Define $f : A \to B$ and $g : B \to C$ as follows

$$\begin{aligned} f(1) &= f(2) = 1 \\ f(3) &= 2 \\ g(1) &= 1 \\ g(2) &= 2 . \end{aligned}$$

Thus, f is onto, g is one-to-one, and the composition $g \circ f$, whose values are shown below is clearly neither one-to-one nor onto.

$$\begin{aligned} g \circ f(1) &= 1 \\ g \circ f(2) &= 1 \\ g \circ f(3) &= 2 \end{aligned}$$

3. (30) Defne a relation R on the set Z of integers as follows

$$mRn \text{ means there is an integer } k \text{ such that } m - n = 3k .$$

(a) Show this is an equivalence relation.

We need to show that the relation R is reflexive, symmetric and transitive.

To see that R is reflexive, note that for any integer m we have $m - m = 3 \cdot 0$. Thus, $mRm \; \forall m \in Z$.

Symmetry follows from the fact that if $k \in Z$ so is $-k$. Suppose mRn. Then there is a $k \in Z$ such that $m - n = 3k$. This implies $n - m = 3(-k)$. Hence nRm.

To verify that R is transitive suppose mRn and nRp. Then there are integers k_1 and k_2 such that

$$\begin{aligned} m - n &= 3k_1 \\ n - p &= 3k_2 . \end{aligned}$$

Adding these two equations we get

$$m - p = 3(k_1 + k_2) .$$

Thus, mRp.

(b) There are three different equialence classes, and they are

$$[0], \; [1], \; [2] .$$

To see this note that if we divide any integer by 3 the remainder must either be 0, 1, or 2. That is,

$$m = 3k + r ,$$

where $r = 0$, 1, or 2. Thus, any integer m is related to 0, 1, or 2. Moreover, these three equivalence classes are mutually disjoint.

(c) If $[p]$ and $[q]$ are two equivalence classes, is the following binary operation well defined?

$$[p] + [q] = [p + q]$$

The answer is yes, it is well defined. To see this suppose $p_1 R p_2$ and $q_1 R q_2$. Then there are integers k_1 and k_2 such that

$$\begin{aligned} p_1 - p_2 &= 3k_1 \\ q_1 - q_2 &= 3k_2 . \end{aligned}$$

Adding these two equations we get

$$p_1 + q_1 - (p_2 + q_2) = 3(k_1 + k_2) .$$

That is, $(p_1 + q_1) R (p_2 + q_2)$. This of course means $[p_1 + q_1] = [p_2 + q_2]$, and the binary operation is well defined on equivalence classes.

4. (10) Define what it means to say that a set A is infinite, and show that if A is infinite and $A \subseteq B$, then B is infinite.

A set A is infinite if there is a function $f : A \to A$ such that f is one-to-one, and not onto.

Suppose now that A is infinite and $A \subseteq B$. Then there is a function $f : A \to A$ such that f is one-to-one, and not onto. Define the function $g : B \to B$ as follows

$$g(x) = \begin{cases} f(x), & \text{if } x \in A \\ x, & \text{if } x \in B - A \end{cases} .$$

It is easy to see that g is one-to-one, and since f is not onto, g cannot be onto either.