1. (20) Define the following:

a. Let A and B be sets, what is $A - B$?

$$A - B = \{x \in A : x \notin B\} .$$

b. Let f be a function from A to B. If $X \in P(A)$, what is $f(X)$?

$$f(X) = \{f(x) : x \in X\} .$$

c. What does it mean to say that a function is onto?

A function f from A to B is onto if the image of f is all of B. That is,

$$\{f(x) : x \in A\} = B$$

d. What is the truth table for $P \Rightarrow Q$?

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

2. (15) Let A, B, and C denote subsets of a universal set U. Determine the truth or falsity of the following equations. If true supply a proof, and if false supply a counter example.

a. $(A - B) \cup (B - A) = A \cup B - A \cap B$

This equality is true. To see that this is so, we'll show that each of the sides of the equality is a subset of the other side. Suppose that $x \in (A - B) \cup (B - A)$. Then $x \in A - B$ or $x \in B - A$. In either case $x \in A \cup B$. Moreover x cannot be in both A and B for if it is in both of these sets it cannot be in $A - B$ or $B - A$. Thus, $x \in A \cap B$ and we have $x \in A \cup B - A \cap B$. Conversely suppose $x \in A \cup B - A \cap B$, Then $x \in A \cup B$ and $x \notin A \cap B$. Thus, if $x \in A$ then $x \notin B$ and we have $x \in A - B$. If $x \in B$, then $x \notin A$ and we have $x \in B - A$. So $x \in (A - B) \cup (B - A)$, and the two sets are equal.

b. $(A - B) - C = A - (B - C)$

These two sets in general are not equal. For an example of this set

$$A = C = \{1\}, B = \{2\} .$$

Then we have

$$(A - B) - C = \emptyset$$

$$A - (B - C) = \{1\} .$$

c. $\overline{A \cup B} = \overline{A} \cap \overline{B}$, where $\overline{A}$ denotes the complement of A in U. That is, $\overline{A} = U - A$.

This equality is true. Suppose $x \in \overline{A \cup B}$. Then $x \notin A \cup B$. That is $x \notin A$ and $x \notin B$, or $x \in \overline{A}$, and $x \in \overline{B}$, or $x \in \overline{A} \cap \overline{B}$. Conversely suppose $x \in \overline{A} \cap \overline{B}$. Then $x \notin A$ and $x \notin B$. Hence $x \notin A \cup B$, or $x \in \overline{A \cup B}$.

3. (10)Prove or disprove the following meta statement

$$(P \Rightarrow Q) \leftrightarrow (P \vee Q),$$

where P and Q are meta statements.

This is not a true statement as can be seen by looking at the truth table for each statement.

P	Q	$P \Rightarrow Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

.

Since the truth values for these two statements are not the same regardless of the truth values of P and Q, the statements are not logically equivalent.

4. (15) A function $f : R \to R$ is said to have the Dizzy Blond Property at the point x_0 if the following is true

$\exists$ real numbers y_1 and y_2, $y_1 \neq y_2$such that
$\forall \epsilon > 0, \exists \delta > 0, \exists$ real numbers x_1, and x_2, such that
$|x_1 - x_0| < \delta$, $|x_2 - x_0| < \delta$, $|f(x_1) - y_1| < \epsilon$, and $|f(x_2) - y_2| < \epsilon$

a. Write out the negation of the Dizzy Blond Property.

$\forall$ real numbers y_1 and y_2, $y_1 \neq y_2$
$\exists \epsilon > 0$, such that $\forall \delta > 0$, $\forall$ real numbers x_1, and x_2
$|x_1 - x_0| \geq \delta$, or $|x_2 - x_0| \geq \delta$, or $|f(x_1) - y_1| \geq \epsilon$, or $|f(x_2) - y_2| \geq \epsilon$

b. Find a function with the Dizzy Blond Property at the point $x_0 = 2$.

Let

$$f(x) = \begin{cases} -1, & x \leq 2 \\ 1, & x > 2 \end{cases}.$$

Set $y_1 = -1$ and $y_2 = 1$. Regardless of the value of ϵ, for any $\delta > 0$, set $x_1 = 2 - \delta/2$ and $x_2 = 2 + \delta/2$. Then we have

$$|x_1 - 2| = |x_2 - 2| = \delta/2 < \delta$$

and

$$|f(x_1) - (-1)| = |f(x_2) - (1)| = 0 < \epsilon.$$

A better definition of the Dizzy Blond Property is:

$\exists$ real numbers y_1 and y_2, $y_1 \neq y_2$such that
$\forall \epsilon > 0, \forall \delta > 0, \exists$ real numbers x_1, and x_2, such that
$|x_1 - x_0| < \delta$, $|x_2 - x_0| < \delta$, $|f(x_1) - y_1| < \epsilon$, and $|f(x_2) - y_2| < \epsilon$

5. (15) Suppose $f : A \to B$ and $g : B \to C$. Determine the truth or falsity of the following equations. If true supply a proof, and if false supply a counter example.

a. If f is one-to-one, then so is $g \circ f$.

This is false. Let $A = \{1, 2\}$, $B = \{1, 2\}$, $C = \{a\}$, with $f(1) = 1, f(2) = 2$, and $g(x) = a$. Then f is one-to-one, and $g \circ f$ is not.

b. If f is onto then so is $g \circ f$.

This too is false. Let $A = \{1, 2\}$, $B = \{1, 2\}$, $C = \{a, b\}$. Define f and g as in part a. Then f is onto, while $g \circ f$ is not.

c. If $g \circ f$. is one-to-one then so is f.

This is true. Suppose $f(x_1) = f(x_2)$. Then $g(f(x_1)) = g(f(x_2))$, or $g \circ f(x_1) = g \circ f(x_2)$. Since $g \circ f$ is one-to-one, we must have $x_1 = x_2$. Thus, f is one-to-one.