**1**. (15) Define or state the following:

**a**. Integers $a$ and $b$ are congruent modulo $n$,

$a$ and $b$ are congruent modulo $n$ if $n$ divides their difference.

**b**. $d$ is the greatest common divisor of two integers $a$ and $b$,

$d$ is the greatest common divisor of $a$ and $b$ if it is a positive integer with the following two properties:

$d$ divides $a$ and $b$

if $k$ divides $a$ and $b$, then $k$ divides $d$ .

**c**. Second principle of induction.

If $S$ is a subset of the natural numbers, $N$, with the following two properties,

$$1 \in S$$

$$\{1, 2, \cdots, n\} \in S \Rightarrow n + 1 \in S,$$

then $S = N$.

**2**. (10) Let $f_n$ be a sequence of natural numbers such that

$$f_1 = 1$$

$$f_n = 2f_{n-1} + 1 \text{ for } n = 2, 3, \cdots .$$

Show that $f_n = 2^n - 1$ for all natural numbers $n$.

Let $S = \{n \in N \ : \ f_n = 2^n - 1\}$. We note that $1 \in S$. Suppose $n \in S$. Then

$$\begin{aligned} f_{n+1} &= 2f_n + 1 = 2(2^n - 1) + 1 \\ &= 2^{n+1} - 1 . \end{aligned}$$

Thus, $n + 1 \in S$. The set $S$ satisfies the conditions of the first principle of induction, and $S$ is therefore equal to the entire set of natural numbers. Thus, $f_n = 2^n - 1$ for all natural numbers.

**3**. (10) How many different 1, 2, 3, or 4 letter words can be made from the letters in "Michael".

Notice first that all of the letters in "Michael" are different. Thus, the number of such words equals

$$\begin{aligned} \binom{7}{1} + \binom{7}{2}2! + \binom{7}{3}3! + \binom{7}{4}4! &= 7 + 21 \cdot 2! + 35 \cdot 3! + 35 \cdot 4! \\ &= 7 + 42 + 210 + 840 \\ &= 1099 . \end{aligned}$$

**4.** (15) Let $m = 90$ and $n = 420$.

**a.** Use the Euclidean algorithm to find the greatest common divisor, $\gcd(m, n)$, of $m$ and $n$, and then use your work to write the gcd as an integer linear combination of $m$ and $n$.

$$\begin{aligned} 420 &= 4 \cdot 90 + 60 \\ 90 &= 60 + 30 \\ 60 &= 2 \cdot 30 \,. \end{aligned}$$

Thus, $\gcd(420, 90) = 30$, and we have

$$\begin{aligned} 30 &= 90 - 60 \\ &= 90 - (420 - 4 \cdot 90) \\ &= 5 \cdot 90 - 420 \,. \end{aligned}$$

**b.** Find the least common multiple of $m$ and $n$.

The easiest way to do this is to use the fact that the product of the gcd and lcm of two integers equals the product of those two integers. Thus,

$$\begin{aligned} \operatorname{lcm}(420, 90) &= \frac{420 \cdot 90}{\gcd(420, 90)} \\ &= \frac{420 \cdot 90}{30} = 420 \cdot 3 \\ &= 1260 \,. \end{aligned}$$

**5.** (15) Suppose that $R$ is an equivalence relation on the set $A$. Let $[a]$ and $[b]$ be two equivalence classes defined by $R$. Show that either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Suppose $[a] \cap [b] \neq \emptyset$. Let $x_0 \in [a] \cap [b]$. To see that $[a] \sqsubseteq [b]$, let $x \in [a]$. Then $xRa$ and $x_0Ra$. Symmetry and transitivity imply that $xRx_0$, and since $x_0 \in [b]$, we deduce that $xRb$, which means $x \in [b]$. Thus, $[a] \sqsubseteq [b]$, a similar argument shows that $[b] \sqsubseteq [a]$. Hence the two equivalence classes are equal.

**6.** (10) Find all integer solutions of the equation

$$6x \equiv 5 (\operatorname{mod} 17) \,.$$

The greatest common divisor of 6 and 17 is 1, and we have

$$1 = 3 \cdot 6 - 17 \,.$$

Thus, $3 \cdot 6 \equiv 1 (\operatorname{mod} 17)$, and we have

$$\begin{aligned} 3 \cdot 6x &\equiv 3 \cdot 5 (\operatorname{mod} 17) \\ x &\equiv 15 (\operatorname{mod} 17) \,. \end{aligned}$$

Thus, $x = 15 + 17k$ for any $k \in Z$ is an integer solution of the equation.