Math 220 - 903 Final Exam Solutions Dec 9, 2008

1. (20) Let U be a universal set, and let a and b be arbitrary subsets of U.

a. What does it mean to say that the sets a and b have the same cardinal number, that is they are numerically equivalent?

a and b have the same cardinal number if there is a one-to-one function from a onto b, or vice-versa.

b. Define a relation R on $P(U)$, the set of subsets of U, by saying a and b are related if they have the same cardinal number. Show that R is an equivalence relation.

To see that R is reflexive suppose $a \in P(U)$. Then $i_a : a \to a$ ($i_a(x) = x \; \forall x \in a$) is a one-to-one map of a onto a. Symmetry is easy for if $f : a \to b$ is a one-to-one map of a onto b, then f^{-1} is a one-to-one map of b onto a. Thus, if aRb then bRa. If a and b have the same cardinal number, and b and c have the same cardinal number, then there are functions $f : a \to b$ and $g : b \to c$, which are both one-to -one and onto. Then $g \circ f$ is a one-to-one function from a onto c. That is, a and c have the same cardinal number.

c. If $U = \{1, 2, 3, 4\}$, what is the equivalence class generated by the set $a = \{1, 2\}$?

There are $\binom{4}{2} = \dfrac{4 \cdot 3}{2} = 6$ subsets of U with cardinal number 4. These subsets are:

$$\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \}.$$

The equivalence class generated by $\{1, 2\}$ is the set in the line above.

2. (10) Let $f : A \to B$ and $g : B \to C$. Determine whether each of the following statements is true or false. If true, supply proof, if false supply a counter example.

a. If $g \circ f$ is onto, then f must be onto.

This is not true. Set $A = \{1, 2\}$, $B = \{1, 2, 3\}$, and $C = \{1, 2\}$. Define f and g as follows:

$$f(1) = 1, f(2) = 2$$
$$g(1) = 1, g(2) = 2, g(3) = 2.$$

Then $g \circ f$ is onto, but f is not onto.

b. If $g \circ f$ is one-to-one, then g must be one-to-one.

This one is also false. Same example as above demonstrates this.

3. (15) A function f with domain $[0, 1]$ and codomain the real numbers is said to be happy if for every $\epsilon > 0$ there is a $\delta > 0$ such that for all x, y in $[0, 1]$ if $|x - y| < \delta$, then $|f(x) + f(y)| < \epsilon$.

a. A function is said to be unhappy if it is not happy. Write the negation of a happy function. That is, what does it mean to say a function is unhappy?

Taking the negation of the definition of a happy function we have: $\exists \epsilon > 0$ such that $\forall \delta > 0 \; \exists x$ and y in $[0, 1]$ such that $|x - y| < \delta$ and $|f(x) + f(y)| \geq \epsilon$.

b. Find an example of a happy function.

The simplest, and the only, example of a happy function is $f(x) = 0$ for all x.

4. (15) Let a and b be integers.

a. Show that if there are integers x and y such that

$$ax + by = 1,$$

then 1 is the greatest common divisor of a and b.

The integer 1 divides both a and b, and if c divides a and c divides b, then it must divide $ax + by$. That is, c divides 1. Thus, 1 is the greatest common divisor of a and b.

b. Suppose there are integers x, y, and d, with $d > 0$, such that

$$ax + by = d\,.$$

Is d the greatest common divisor of a and b?

The answer is no. Suppose d_0 is the greatest common divisor of a and b. Then there are integers x_0 and y_0 such that

$$ax_0 + by_0 = d_0\,.$$

If we multiply this equation by any non-zero positive integer α we have

$$a(\alpha x_0) + b(\alpha y_0) = \alpha d_0\,.$$

But αd_0 is not the greatest common divisor of a and b.

5. (15) Suppose m and n are relatively prime and both divide the integer c. Show that their product mn must also divide c.

Since m and n both divide c, there are integers k and l such that $c = mk = nl$. This says, for example that m divides nl, but m and n are relatively prime so this means that m divides l. Thus, there is an integer r such that $rm = l$, and this gives us $c = nl = nrm$. Hence mn divides c.

6. (15) State the Well Ordering Principle (least element principle) of the natural numbers, and use it to prove the first principle of induction.

The Well Ordering Principle states that any non-empty subset of the natural numbers has a least element.

Let S be any subset of the natural numbers, which has the following two properties:

$$1 \in S$$
$$n \in S \Rightarrow n + 1 \in S\,.$$

Let $T = N - S$. If T is not empty it, by the Well Ordering Principle, must have a least element n_0. This element is not 1 as $1 \in S$. Then $n_0 - 1$ is a natural number not in T. That is, $n_0 - 1 \in S$, but then property two above says that $n_0 = (n_0 - 1) + 1 \in S$, but this is impossible as $T = N - S$. Thus, the set T must be the empty set, or S is the set of natural numbers, which means we've shown that the first principle of induction is a consequence of the Well Ordering Principle.

7. (10) For every natural number n show that 3 divides $4^{n+1} + 5^{2n-1}$.

If $n = 1$, we have $4^2 + 5^1 = 16 + 5 = 21$, which is divisible by 3. Now suppose that 3 divides $4^{n+1} + 5^{2n-1}$. Then

$$\begin{aligned} 4^{(n+1)+1} + 5^{2(n+1)-1} &= 4 \cdot 4^{n+1} + 5^2 \cdot 5^{2n-1} \\ &= 4 \cdot 4^{n+1} + 4 \cdot 5^{2n-1} - 4 \cdot 5^{2n-1} + 5^2 \cdot 5^{2n-1} \\ &= 4(4^{n+1} + 5^{2n-1}) + 5^{2n-1}(25 - 4)\,. \end{aligned}$$

By the induction assumption 3 divides the first summand and 3 also divides $21 = 25 - 4$. Thus, 3 divides the sum of the two terms, and we conclude by the first principle of induction that 3 divides $4^{n+1} + 5^{2n-1}$ for all natural numbers n.