1. (30) Define the following:

(a) If A and B are two sets, what is $A - B$?

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

(b) If A and B are two sets, what is $A \cup B$?

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

(c) If $f : A \to B$, and $X \subseteq A$, what is $f(X)$?

$$\begin{aligned} f(X) &= \{f(a) : a \in X\} \\ &= \{y \in B : \exists\, a \in X \text{ such that } f(a) = y\} \end{aligned}$$

(d) If $f : A \to B$, and $Y \subseteq B$, what is $f^{-1}(Y)$?

$$f^{-1}(Y) = \{x \in A : f(x) \in Y\}$$

2. (30) A function $f : [a, b] \to R$ is said to be continuous at a point $c \in [a, b]$ if

$$\begin{gathered} \forall\, \epsilon > 0,\ \ \exists \delta > 0, \text{ such that } \forall x \\ \text{if } x \in [a, b] \text{ and } |x - c| < \delta, \text{ then } |f(x) - f(c)| < \epsilon. \end{gathered}$$

(a) Write the negation of this statement.

The easiest way to write the negation is to first rewrite $P \to Q$ as $(-P) \vee Q$. Then negating $(P \to Q)$ is easily done. The negation of the above statement is

$$\begin{gathered} \exists\, \epsilon > 0, \forall\, \delta > 0\ \exists\, x \text{ such that} \\ -\left((x \in [a, b] \ \wedge\ |x - c| < \delta) \to |f(x) - f(c)| < \epsilon\right) \end{gathered}$$

or

$$\begin{gathered} \exists\, \epsilon > 0, \forall\, \delta > 0, \exists\, x \ \text{ such that} \\ (x \in [a, b] \ \wedge\ |x - c| < \delta) \wedge (|f(x) - f(c)| \geq \epsilon) \end{gathered}$$

(b) If $[a, b] = [0, 1]$, give an example of a real valued function defined on the interval $[0, 1]$ that is not continuous at $c = 0$. Be sure to show that your function satisfies the statement you gave for part a.

Define $f : [0, 1] \to R$ by $f(x) = \begin{cases} 0, & x = 0 \\ 1, & 0 < x \leq 1 \end{cases}$. To see that this function satisfies the negation of continuity given in part a. Set $\epsilon = 1/2$. Then for any $\delta > 0$ set $x = \min\{1,\ \delta/2\}$. Then $x \in [0, 1]$ and $|x - 0| < \delta$, and

$$|f(x) - f(0)| = |1 - 0| = 1 > 1/2.$$

3. (30) Determine which of the following statements are true and which are false. If a statement is true supply a proof, and if it's false supply a counter example.

   (a) For any two sets A and B, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

   This is not a true statement. There are many possible counter examples. In the following counter example, the universal set is the set of positive integers. Set $A = \{1, 2\}$ and $B = \{3, 4\}$. Then $\overline{A \cup B} = \{5, 6, \cdots\}$, $\overline{A} = \{3, 4, \cdots\}$, and $\overline{B} = \{1, 2, 5, 6, \cdots\}$. Thus, $\overline{A} \cup \overline{B} = \{1, 2, \cdots\}$, and this is not equal to $\overline{A \cup B}$.

   (b) The logical statement $[(-Q) \wedge (P \to Q)] \to (-P)$ is a tautology.

   This is true, and a truth table is the most straightforward way to see this.

4. (10) Let $f : A \to B$. Prove or disprove

$$\overline{f^{-1}(C)} = f^{-1}\left(\overline{C}\right),$$

for every subset C of B. Note: The complement on the left hand side of the equals sign is with respect to A, while the complement on the right hand side of the equals sign is with respect to B.

The equality is true. To see this is the case, we'll show that each of the sets is a subset of the other. To this end suppose that $x \in \overline{f^{-1}(C)}$, then $x \notin f^{-1}(C)$. This implies $f(x) \notin C$. Thus $f(x) \in \overline{C}$. Hence $x \in f^{-1}\left(\overline{C}\right)$. Now suppose that $x \in f^{-1}\left(\overline{C}\right)$. Then

$f(x) \in \overline{C}$. Hence $f(x) \notin C$.This says that $x \notin f^{-1}(C)$ or $x \in \overline{f^{-1}(C)}$. Thus, the two sets are equal.