1. (25) Let $f : A \to B$. Suppose X_1 and X_2 are subsets of A, and Y_1 and Y_2 are subsets of B. Determine whether or not the statements below are true or false. If true, supply a proof, and if false, provide a counter example.

   (a) $f(X_1 \cup X_2) = f(X_1) \cup f(X_2)$

   This is a true statement. To verify that it is true suppose $y \in f(X_1 \cup X_2)$. Then there is an $x \in X_1 \cup X_2$ such that $f(x) = y$. Now $x \in X_1$ or $x \in X_2$. Thus, $y = f(x) \in f(X_1)$ or $y \in f(X_2)$. Thus, $y \in f(X_1) \cup f(X_2)$. Conversely suppose $y \in f(X_1) \cup f(X_2)$. Then $y \in f(X_1)$ or $y \in f(X_2)$. If $y \in f(X_1)$, then there is an $x \in X_1$ such that $f(x) = y$. But then $x \in X_1 \cup X_2$ and $y = f(x) \in f(X_1 \cup X_2)$. A similar statement is true if $x \in X_2$. Since both sides of the equality have been shown to be subsets of each other, the two sets are equal.

   (b) $f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2)$

   This is also a true statement. Suppose $x \in f^{-1}(Y_1 \cup Y_2)$. Then $f(x) \in Y_1 \cup Y_2$. Hence $f(x) \in Y_1$ or $f(x) \in Y_2$, or $x \in f^{-1}(Y_1)$ or $x \in f^{-1}(Y_2)$. Thus, $x \in f^{-1}(Y_1) \cup f^{-1}(Y_2)$. Now suppose that $x \in f^{-1}(Y_1) \cup f^{-1}(Y_2)$. Then $x \in f^{-1}(Y_1)$ or $x \in f^{-1}(Y_2)$. This means that $f(x) \in Y_1$ or $f(x) \in Y_2$. Hence $f(x) \in Y_1 \cup Y_2$, or $x \in f^{-1}(Y_1 \cup Y_2)$. This shows that the two sets are equal, since each is a subset of the other.

2. (25) Let $f : A \to R$ be defined by $f(x) = x^3 - x$, where A is defined below.

(a) If the domain A of f is the set R of all real numbers, show that f is not one-to-one.

The easiest way to see that f is not one-to-one is to exhibit two or more values of x for which $f(x)$ has the same value. Note that $x^3 - x = x\left(x^2 - 1\right)$. This expression has three distinct roots. That is, three different values of x for which $f(x) = 0$. They are -1, 0, and 1.

(b) If the domain A of f is the set $\{x : \ x \geq 1\}$, show that f is one-to-one.

Suppse x_1 and x_2 are both greater than or equal to one, that they are not equal, and

$$x_1^3 - x_1 = x_2^3 - x_2 \ .$$

Then the following equations are true

$$\begin{aligned} x_1^3 - x_2^3 &= x_1 - x_2 \\ (x_1 - x_2)\left(x_1^2 + x_1x_2 + x_2^2\right) &= x_1 - x_2 \\ x_1^2 + x_1x_2 + x_2^2 &= 1 \ . \end{aligned}$$

However, this last equation cannot hold. Each of x_1 and x_2 is no smaller than 1, which implies $x_1^2 + x_1x_2 + x_2^2 \geq 3$, a clear contradiction of the last equality. Hence if $f(x_1) = f(x_2)$ we must have $x_1 = x_2$.

(c) If the domain A of f is the set $\{x : \ x \geq 1\}$, what is the image of f ?

We note that $f(1) = 0$, that $f\ '(x) = 3x^2 - 1 > 0$ on the set A. Thus, f is an increasing function, with limit ∞ as x tends to ∞. Since f is continuous, the image of f must contain every real number greater than or equal to zero. That is, the image of f equals $[0, \infty)$.

(d) Let B denote your answer to part c. Thus, $f : \{x : x \geq 1\} \to B$ is one-to-one and onto, and has an inverse function. Let $f^{\ -1}$ denote this inverse function. Which of the following are true?

i. $f^{\ -1}(0) = -1$

This is not true, since -1 does not belong to the domain of f.

ii. $f^{\ -1}(4) = 2$

If this was true then $f(2)$ must equal 4. However,

$$f(2) = 2^3 - 2 = 6 \ .$$

So $f^{\ -1}(4) \neq 2$. Moreover, this calculation shows that $f^{\ -1}(6) = 2$.

iii. $f^{\ -1}(6) = 2$

This is true as we saw in part ii.

3. (25) Let the collection of sets $\{P_i\}_{i\in I}$ be a partition of the set A. That is, no set P_i is the empty set, and

$$\begin{aligned} P_i \cap P_j &= \emptyset, \text{ if } i \neq j \\ \cup_{i\in I} P_i &= A \end{aligned}$$

Use this partition to define a relation R on the set A that is an equivalence relation such that the equivalence classed defined by R are the same sets P_i we started with. Be sure to prove your claims.

Define a relation R on the set A by aRb if and only if a and b are in the same set P_i. To see that R is an equivalence relation we need to show that it is reflexive, symmetric, and transitive. Since $\cup_{i\in I} P_i = A$, we know that for any $a \in A$ there is an i such that $a \in P_i$. Thus, aRa. Symmetry is obvious. For if aRb, then for some i a and b are in the set P_i, but then b and a are in the set P_i, and we have bRa. Suppose next that aRb and bRc. Then there is an i and j such that a and b are in P_i, and b and c are in P_j. This means that $b \in P_i \cap P_j$. Hence, since the collection of sets $\{P_i\}$ are mutually disjoint, we have $P_i = P_j$. Thus, a and c are in the same set, and we have aRc. This shows that R is an equivalence relation.

Now let $[a]$ denote the equivalence class generated by the relation R and the element a. Since $[a] = \{b \in A : aRb\} = \{b \in A : a \text{ and } b \text{ are in the same set } P_i\} = P_i$, where P_i is that one set, amongst all of the partitioning sets, that contains the point a. Thus, the equivalence classes are the same sets used to define the equivalence relation R.

4. (25) Let R be a relation on Z defined by mRn if $m - n = 3k$ for some integer k.

   (a) Show that R is an equivalence relation.

   R is reflexive, since for any integer n, we have $n - n = 3 \cdot 0$. Thus, nRn. To see that R is symmetric suppose that nRm, then we have $n - m = 3k$ or $m - n = 3(-k)$. Thus, mRn. Finally suppose that mRn and nRp. That is, there are integers k_1 and k_2 such that

   $$\begin{aligned} m - n &= 3k_1 \\ n - p &= 3k_2 \end{aligned}$$

   Adding these two equations together we get

   $$m - p = 3(k_1 + k_2).$$

   Hence mRp and we see that R is an equivalence relation.

   (b) How many different equivalence classes does R define?

   There are 3 different equivalence classes. They are

   $$[0], [1], \text{ and } [2] .$$

   The easiest way to see this is the following: let n be any integer, than by the division algoritm we have integers q and r such that

   $$n = 3q + r ,$$

   where $r = 0$, 1, or 2. Thus, for any integer n, the number $n - r$ is divisible by 3, for r equal to one of 0, 1, or 2. This means that n belongs to one of the three equivalence classes [0], [1], or [2]. Moreover, it is clear that $r_1 - r_2$ can never be divisible by three when the r_i's are 0, 1, or 2, unless they are equal. Thus, the three listed equivalence classes are all different.