1. (15) Define the following:

   (a) a transitive relation on a set A. Note: there are two terms that need to be defined. They are *relation* and *transitive.*

   First a relation on a set A is just a subset of $A \times A$ the Cartesian product of A with itself. To say that a relation R is transitive is to say that if xRy and yRz, then xRz.

   (b) f is a bijection from A to B.

   A bijection from a set A to B is a one-to-one function with domain A and codomain B that is also onto.

   (c) $*$ is an associative binary operation on a set A. Note: there are two terms that need to be defined. They are *binary operation* and *associative.*

   A binary operation is a mapping from $A \times A$ into A. The mapping (operation) is said to be associative if

$$(a * b) * c = a * (b * c),$$

   for all a, b, and c in the set A.

2. (20) Let $f : A \to B$. Suppose X_1 and X_2 are subsets of A, and Y_1 and Y_2 are subsets of B. Determine whether or not the statements below are true or false. If true, supply a proof, and if false, provide a counter example.

   (a) $f(X_1 \cap X_2) = f(X_1) \cap f(X_2)$

   This equation is not true in general. A counter example is given by the following: $A = \{a, b, c\} = B$, with $f(a) = a$, $f(b) = b$, and $f(c) = a$. Set $X_1 = \{a, b\}$ and $X_2 = \{b, c\}$. Then we have

$$\begin{aligned} f(X_1 \cap X_2) &= f(\{b\}) = \{b\} \\ f(X_1) &= f(\{a, b\}) = \{a, b\} \\ f(X_2) &= f(\{b, c\}) = \{a, b\}. \end{aligned}$$

   Thus, we have $f(X_1 \cap X_2) \neq f(X_1) \cap f(X_2)$. It is true, however, that $f(X_1 \cap X_2) \sqsubseteq f(X_1) \cap f(X_2)$ is always true.

   (b) $f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$

   This equality of true. Suppose that $x \in f^{-1}(Y_1 \cap Y_2)$, then $f(x) \in Y_1 \cap Y_2$. Thus, $x \in f^{-1}(Y_1)$ and $x \in f^{-1}(Y_2)$, and therefore $x \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$. So the left hand side is a subset of the right hand side. Now suppose that $x \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$. Then $f(x) \in Y_1$ and $f(x) \in Y_2$. Thus, $f(x) \in Y_1 \cap Y_2$. So $x \in f^{-1}(Y_1 \cap Y_2)$, and the right hand side is a subset of the left hand side. Therefore the two sets are equal.

3. (15) Let R be an equivalence relation on a set A. Let $[a]$ denote the equivalence class generated by the element $a \in A$. Show that for any two elements a and b in A either

$$[a] \cap [b] = \emptyset \text{ or } [a] = [b] .$$

First we remember that $[a] = \{x \in A : aRx\}$. Suppose that the two equivalence classes are not disjoint. Let $x \in [a] \cap [b]$. Thus, aRx and bRx. Note: the symmetry and transitivity of the relation R imply that aRb.

Let $c \in [a]$. Then aRc, and the symmetry of R implies that cRa. Since R is transitive cRa and aRb imply that cRb, which, by symmetry, implies bRc or $c \in [b]$. A similar argument shows that if $c \in [b]$, then $c \in [a]$. Thus, we have $[a] = [b]$.

4. (15) Let R be a relation on R^2 defined by $(x, y)\, R\, (u, v)$ if $y - v = 3\,(x - u)$.

   (a) Show that R is an equivalence relation.

   We have to show that R is reflexive, symmetric, and transitive.

   So suppose $(x, y) \in R^2$. Then it is certainly true that $y - y = 3\,(x - x)$. Thus, $(x, y)\, R\, (x, y)$, and R is reflexive.

   To see that R is symmetric, suppose that $(x, y)\, R\, (u, v)$. Then $y - v = 3\,(x - u)$, or $v - y = 3\,(u - x)$. Hence $(u, v)\, R\, (x, y)$.

   For transitivity suppose that $(x, y)\, R\, (u, v)$ and $(u, v)\, R\, (s, t)$. Then we have

$$\begin{aligned} y - v &= 3\,(x - y) \text{ and} \\ v - t &= 3\,(u - s) . \end{aligned}$$

   Adding these two equations together we get

$$y - t = 3\,(x - s) .$$

   This means that $(x, y)\, R\, (s, t)$, and we've shown that R is indeed transitive. Thus, since R is reflexive, symmetric, and transitive, R is an equivalence relation.

   (b) What do the equivalence classes of R look like as subsets of R^2?

   The equivalence classes are straight lines with slope equal to 3.

5. (15) A set A is said to be infinite if there is a function $f : A \to A$ that is one-to-one and not onto.

(a) Show that the set of even natural numbers $E = \{2, 4, 6, \cdots\}$ is infinite.

Define $f : E \to E$ by $f(x) = 2x$. For every even integer x, $2x$ is also an even integer. Thus f has domain E and codomain E. To see that f is one-to-one, suppose that $f(x_1) = f(x_2)$, then $2x_1 = 2x_2$, which implies that $x_1 = x_2$. Thus, f is one-to-one. The function f is not onto E as there is no $x \in E$ such that $f(x) = 2 \in E$. For if $f(x) = 2$, then $x = 1 \notin E$. Since we've exhibited a one-to-one mapping of E into itself that is not onto, the set E is infinite.

(b) Show that the set $A = \{1, 2\}$ is not infinite.

One way to do this is to show that every one-to-one mapping of A into itself must be onto. So suppose that $f : A \to A$ is one-to-one. Either $f(1) = 1$ or $f(1) = 2$. Suppose that $f(1) = 1$. Then, since f is one-to-one $f(2) \neq 1$, so $f(2) = 2$. Thus, f is onto. A similar argument shows that if $f(1) = 2$, then $f(2) = 1$, and f is again onto. So, there cannot exist a mapping from A into itself that is both one-to-one and onto.

(c) Show that if A is infinite and $A \sqsubseteq B$, then B is infinite.

Since A is infinite $\exists f$ such that $f : A \to A$ that is one-to-one and not onto. Let $X = A - f(A)$. Then $X \neq \emptyset$. Define $g : B \to B$ as follows

$$g(b) = \begin{cases} f(b) & b \in A \\ b & b \in B - A \end{cases}.$$

Since the set $X \sqsubseteq A$, the image of g cannot intersect this set. Thus, g is not onto. To see that g is one-to-one, suppose that $g(x) = g(y)$. There are three cases to consider, both x and y are in A or they are both in $B - A$, or one of them is in A and the other is in $B - A$.

Case (both in A): in this case $g(x) = g(y)$ implies that $f(x) = f(y)$. Since f is one-to-one, $x = y$.

Case (both in $B - A$): here by definition of g, $x = g(x) = g(y) = y$.

Case (one in A and the other in $B - A$): suppose $x \in A$, then $g(x) = f(x) \in A$ too, and $y \in B - A$ implies $g(y) = y \in B - A$, and this of course cannot happen, since A and $B - A$ are disjoint. Thus, the function g is one-to-one, and B must be an infinite set.

6. (20) Define the following binary operation on the set Z of integers

$$m * n = m + n - mn$$

(a) Show that $*$ is indeed a binary operation.

By definition a binary operation on Z is a mapping from $Z \times Z$ into Z. $*$ is certainly defined on $Z \times Z$, and, for any two integers m and n, it is certainly true that $m+n-mn$ is an integer. Thus, $m * n \in Z$.

(b) Is $*$ associative?

This binary operation is indeed associative. For

$$\begin{aligned}
(m * n) * p &= (m + n - mn) * p \\
&= (m + n - mn) + p - (m + n - mn) p \\
&= m + n - mn + p - mp - np + mnp \\
&= m + n + p - np - mn - mp + mnp \\
&= m + (n + p - np) - m (n + p - np) \\
&= m + (n * p) - m (n * p) \\
&= m * (n * p) .
\end{aligned}$$

(c) Is $*$ commutative?

The operation is also commutative.

$$\begin{aligned}
m * n &= m + n - mn = n + m - nm \\
&= n * m
\end{aligned}$$

(d) If we define $*$ on the set, $N \times N$, where N is the set of natural numbers, as above, is it a binary operation on N ?

$*$ is no longer a binary operation as $m * n$ need not be a natural number. A specific example of this is

$$\begin{aligned}
2 * 3 &= 2 + 3 - 2 \cdot 3 \\
&= 5 - 6 \\
&= -1.
\end{aligned}$$