

1. (20) Let  $A$ ,  $B$ , and  $C$  denote three sets. Prove or disprove the following:

(a)  $(A \cup B) - C = (A - C) \cup (B - C)$

This is a true statement. To see this I'll show that each side of the equality is a subset of the other side. So suppose  $x \in (A \cup B) - C$ . Then  $x \in A \cup B$  and  $x \notin C$ . Thus,  $x \in A$  or  $x \in B$ . Hence,  $x \in A - C$  or  $x \in B - C$ , and we have  $x \in (A - C) \cup (B - C)$ . Now suppose  $x \in (A - C) \cup (B - C)$ . Then  $x \in A - C$  or  $x \in B - C$ . In either case  $x \notin C$ . Moreover,  $x \in A$  or  $x \in B$ . Thus,  $x \in A \cup B$ , and we have  $x \in (A \cup B) - C$ .

(b)  $(A - B) - C = A - (B - C)$

This identity is not valid in general. To see this set  $A = \{1\}$ ,  $B = \{1\}$ , and  $C = \{1\}$ . Then we have

$$\begin{aligned} (A - B) - C &= (\{1\} - \{1\}) - \{1\} = \emptyset - \{1\} = \emptyset, \text{ and} \\ A - (B - C) &= \{1\} - (\{1\} - \{1\}) = \{1\} - \emptyset = \{1\}. \end{aligned}$$

2. (15) Let  $m$  and  $n$  be any two integers.

(a) Define what it means to say  $m$  divides  $n$ .

$m$  divides  $n$  means that there is an integer  $k$  such that

$$n = km.$$

(b) Show that if  $r|m$  and  $r|n$ , where  $r$  is some integer, then for any integers  $\alpha$  and  $\beta$   $r|(\alpha m + \beta n)$ .

Since  $r|m$  and  $r|n$  there are integers  $k_1$  and  $k_2$  such that

$$\begin{aligned} m &= k_1 r \\ n &= k_2 r. \end{aligned}$$

Thus, we have

$$\alpha m + \beta n = \alpha(k_1 r) + \beta(k_2 r) = (\alpha k_1 + \beta k_2) r,$$

from which we see that  $r|(\alpha m + \beta n)$ .

3. (15) Let  $m$  and  $n$  be any two integers not both zero.

(a) Define the greatest common divisor of  $m$  and  $n$ .

The greatest common divisor of  $m$  and  $n$  is a positive integer  $g$  such that  $g$  divides both  $m$  and  $n$ , and if any other integer divides both  $m$  and  $n$ , it also divides  $g$ .

(b) Suppose  $r_i$  are integers such that  $0 < r_2 < r_1 < m$ , and there are integers  $q_1$ ,  $q_2$ , and  $q_3$  such that

$$\begin{aligned}n &= mq_1 + r_1 \\m &= r_1q_2 + r_2 \\r_1 &= r_2q_3.\end{aligned}$$

Show that  $r_2$  is the greatest common divisor of  $m$  and  $n$ .

To see that  $r_2$  divides both  $m$  and  $n$ , we note that  $r_2$  divides  $r_1$ . So  $r_2$  divides  $r_1q_2 + r_2$ , and this from the second equation equals  $m$ . So  $r_2$  divides  $r_1$  and  $m$ . Thus,  $r_2$  divides  $mq_1 + r_1$ , which, from the first equation, equals  $n$ .

Suppose now that  $g$  divides both  $m$  and  $n$ . Then the first equation tells us that  $g$  divides  $r_1$ , the second equation then tells us that  $g$  divides  $r_2$ . So  $r_2$  must be the greatest common divisor of  $m$  and  $n$ .

(c) Find the greatest common divisor,  $g$ , of 154 and 46. Then find integers  $a$  and  $b$  such that

$$g = 46a + 154b .$$

The Euclidean algorithm is used to calculate  $g$ .

$$\begin{aligned}154 &= 46(3) + 16 \\46 &= 16(2) + 14 \\16 &= 14(1) + 2 \\14 &= 2(7) .\end{aligned}$$

So the greatest common divisor of 154 and 46 is 2. And

$$\begin{aligned}2 &= 16 - 14 \\&= 16 - (46 - 2(16)) = 3(16) - 46 \\&= 3(154 - 3(46)) - 46 \\&= 3(154) - 10(46) .\end{aligned}$$

4. (15) Suppose that  $f : Z^+ \rightarrow Z^+$  satisfies  $f(1) = 1$ ,  $f(2) = 2$  and also satisfies the recursive relation

$$f(n+1) = f(n) + 2f(n-1), \quad n = 2, 3, \dots$$

Prove, by induction, that  $f(n) = 2^{n-1}$  for all  $n \in Z^+$ .

Since  $f(1) = 1 = 2^0 = 2^{1-1}$ , the formula is valid for  $n = 1$ . Now assume it is valid for all positive integers  $k \leq n$ . Then we have

$$\begin{aligned} f(n+1) &= f(n) + 2f(n-1) \\ &= 2^{n-1} + 2(2^{n-2}) \\ &= 2^{n-1} + 2^{n-1} \\ &= 2^n. \end{aligned}$$

By the second principle of induction we see that the formula is valid for all  $n$ .

5. (15) A function  $f : R \rightarrow R$  is said to be bad if  $\exists \epsilon > 0$  such that  $\forall x, \forall \delta > 0, \exists y$  such that

$$|x - y| < \delta \text{ and } |f(x) - f(y)| > \epsilon.$$

- (a) Write the negation, in logical notation, of this definition.

$$\forall \epsilon > 0, \exists x, \exists \delta > 0, \forall y$$

$$|x - y| \geq \delta \text{ or } |f(x) - f(y)| \leq \epsilon.$$

- (b) Show that  $f(x) = x$  is not a bad function.

To see that  $f(x) = x$  is not a bad function we'll verify that  $f$  satisfies the negation of being a bad function. To that end let  $\epsilon > 0$ . Set  $x = 1$  (actually any value of  $x$  will work, but we only need find one  $x$  for a particular  $\epsilon$ ). Set  $\delta = \epsilon$ . Then for any real number  $y$ , if  $|1 - y| \geq \delta$  is not true, then  $|1 - y| < \delta$ , and we must have

$$|f(1) - f(y)| = |1 - y| < \delta = \epsilon.$$

So  $f(x) = x$  is not a bad function.

6. (20) Let  $A = \{(x, y) : x \text{ and } y \text{ are real numbers}\}$ . Define a relation  $R$  on  $A$  by

$$(x_1, y_1) R (x_2, y_2)$$

if there are integers  $m$  and  $n$  such that

$$\begin{aligned}x_2 &= m + x_1 \\y_2 &= n + y_1.\end{aligned}$$

(a) Show that  $R$  is an equivalence relation.

Since  $x = 0 + x$  and  $y = 0 + y$  we have  $(x, y) R (x, y)$ , and  $R$  is reflexive.

Now suppose that  $(x_1, y_1) R (x_2, y_2)$  then we have integers  $m$  and  $n$  such that

$$\begin{aligned}x_2 &= m + x_1 \\y_2 &= n + y_1.\end{aligned}$$

These equations imply that

$$\begin{aligned}x_1 &= -m + x_2 \\y_1 &= -n + y_2.\end{aligned}$$

Which is the statement that  $(x_2, y_2) R (x_1, y_1)$ , and we've demonstrated that  $R$  is symmetric. The last item to verify is that  $R$  is transitive. So suppose that  $(x_1, y_1) R (x_2, y_2)$  and  $(x_2, y_2) R (x_3, y_3)$ . Then there are integers  $m, n, p,$  and  $q$  such that

$$\begin{aligned}x_2 &= m + x_1 \\y_2 &= n + y_1 \\x_3 &= p + x_2 = p + (m + x_1) = (p + m) + x_1 \\y_3 &= q + y_2 = q + (n + y_1) = (q + n) + y_1.\end{aligned}$$

Thus, we see that  $x_3$  is the sum of  $x_1$  and some integer, and  $y_3$  is also the sum of  $y_1$  plus some integer. Hence,  $(x_1, y_1) R (x_3, y_3)$ . Since the relation  $R$  is reflexive, symmetric and transitive it is an equivalence relation.

- (b) Show that for each equivalence class  $[(x, y)]$  there is a unique pair of numbers  $x_1$  and  $y_1$ , where  $0 \leq x_1 < 1$  and  $0 \leq y_1 < 1$  such that

$$[(x, y)] = [(x_1, y_1)] .$$

Let  $x$  and  $y$  be arbitrary real numbers. Let  $m$  and  $n$  be those unique integers such that

$$\begin{aligned} m &\leq x < m + 1 \\ n &\leq y < n + 1. \end{aligned}$$

Set  $x_1 = x - m$  and  $y_1 = y - n$ . Then we have  $0 \leq x_1 < 1$  and  $0 \leq y_1 < 1$  and  $(x, y) R(x_1, y_1)$ . Thus,  $[(x, y)] = [(x_1, y_1)]$ . The fact that  $x_1$  and  $y_1$  are unique follows from the fact that the above  $m$  and  $n$  are uniquely determined by  $x$  and  $y$  respectively. First we show that  $m$ , let's say, is unique, and then show why that means  $x_1$  is unique. A similar argument then shows that  $y_1$  is also unique.

To see that  $m$  is unique suppose we have  $m_1$  and  $m_2$  where

$$m_i \leq x < m_i + 1 \text{ for } i = 1 \text{ and } 2.$$

If the  $m_i$  are not the same then one must be less than the other. Suppose that  $m_1 < m_2$ . Then we have

$$\begin{aligned} -m_1 - 1 &< -x \leq -m_1 \\ m_2 &\leq x < m_2 + 1. \end{aligned}$$

Adding these two together we get

$$m_2 - m_1 - 1 < 0 < 1.$$

However, since  $m_2 - m_1 \geq 1$  We have the following nonsensical statement:

$$0 \leq m_2 - m_1 - 1 < 0 < 1.$$

Clearly  $0 < 0$  cannot be true, implying we cannot have  $m_1 < m_2$ . Hence, the integer  $m$  for which

$$m \leq x < m + 1$$

is unique.

Now suppose that there are two  $x_i$ 's such that  $0 \leq x_i < 1$  and

$$[(x, y)] = [(x_1, y_1)] = [(x_2, y_2)].$$

But this says that there are integers  $m_1$  and  $m_2$  such that

$$\begin{aligned} x &= m_1 + x_1 \\ x &= m_2 + x_2. \end{aligned}$$

This, in turn, implies that

$$\begin{aligned} m_1 &\leq x = m_1 + x_1 < m_1 + 1 \\ m_2 &\leq x = m_2 + x_2 < m_2 + 1 \end{aligned}$$

Hence, we must have  $m_1 = m_2$ , which then gives us

$$x_1 = x - m_1 = x - m_2 = x_2.$$

Similarly  $y_1$  must equal  $y_2$ .