1. (20) Define the following terms:

   (a) Statement (in the context of logic)

   A statement is a declarative sentence, which is either true or false.

   (b) The Cartesian product of two sets A and B.

   The Cartesian product of A and B is

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

   (c) For each $\alpha \in \Lambda$, let A_α be a set. Define $\cap_{\alpha\in\Lambda} A_\alpha$.

   This set is defined as

$$\cap_{\alpha\in\Lambda} A_\alpha = \{x : x \in A_\alpha \text{ for each } \alpha \in \Lambda\ \}$$

   (d) If P and Q are statements, define the statement $P \implies Q$. Note: a truth table is much preferred.

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

2. (15) State Peano's axioms, and give an example of a set that does not satisfy these axioms.

   The axioms are posted on the web page for this course. An example of a set that does not satisfy the axioms is $S = \{a\}$. Since there is not a second element in this set $a + 1$ is not defined, so S does not satisfy the axioms.

3. (20) For each of the statements below decide if they are true or false. If a statement is true prove it, and if it's false supply a counter example.

   (a) $\overline{A \cup B} = A \cap B$.

   This statement is false. An example to demonstrate this is: set the universal set equal to N the natural numbers. Set $A = \{1\}$ and $B = \{2\}$. Then $\overline{A \cup B} = \{3,\ 4,\ \cdots\}$, and $A \cap B = \emptyset$.

   (b) $P \implies Q$ is logically equivalent to $P \wedge (\neg Q)$.

   This statement is false. If you look at the truth tables for these two statements this becomes clear

P	Q	$P \implies Q$	$P \wedge (\neg Q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

4. (15) Use induction to verify the formula

$$\sum_{i=1}^{n} (2i - 1) = n^2.$$

Set $P = \left\{ n \in N : \sum_{i=1}^{n} (2i-1)^2 = n^2 \right\}$. To see that $1 \in P$, we evaluate both sides of the formula with $n = 1$. The RHS equals 1, and the LHS equals

$$\sum_{i=1}^{1} (2i - 1) = (2 - 1) = 1.$$

Thus, $1 \in P$. Assume that $n \in P$. Then we have

$$\begin{aligned} \sum_{i=1}^{n+1} (2i-1) &= \sum_{i=1}^{n} (2i-1) + (2n+1) \\ &= n^2 + 2n + 1 = (n+1)^2 \ . \end{aligned}$$

Thus, $n + 1 \in P$, and the induction axiom tells us that $P = N$.

5. (10) Let A be a set, and let $\mathrm{P}(A)$ denote the power set of A. Let $|A|$ denote the number of elements in the set A.

   (a) If $A = \{1, q, 5\}$, what is $\mathrm{P}(A)$.

   $$\mathrm{P}(A) = \{\emptyset, \{1\}, \{q\}, \{5\}, \{1, q\}, \{1, 5\}, \{5, q\}, \{1, q, 5\}\} \ .$$

   (b) Prove the following formula

   $$|\mathrm{P}(A)| = 2^{|A|} \ .$$

   One way to prove this is by induction. So let

   $$P = \left\{ n \in N : |\mathrm{P}(A)| = 2^{|A|}, \text{ where } |A| = n \right\} \ .$$

   It is easy to see that $1 \in P$. So assume that $n \in P$. Let A be a set with $n + 1$ elements. Denote one of these elements by a. Any subset of A either contains the element a or it doesn't. The number of subsets that do not contain a are the same as the number of subsets of a set with n elements. By the assumption that $n \in P$, there are 2^n such subsets. Any subset of A that contains a can be obtained by adding a to a subset of A that does not contain a; there are 2^n such subsets. Thus, the number of subsets of A equals

   $$|\mathrm{P}(A)| = 2^n + 2^n = 2^{n+1},$$

   and we see that $n + 1 \in P$. By induction axiom we have $P = N$.

6. (20) A function $f(x)$ is said to be ambivalent with respect to l at the point $x = a$ if

$$\begin{aligned} \forall\epsilon &> 0, \forall\delta > 0, \exists\ x_1 \text{ and } \exists\ x_2 \text{ such that} \\ |x_1 - a| &< \delta,\ |x_2 - a| < \delta,\ |f(x_1) - l| < \epsilon \text{ and } |f(x_2) - l| > \epsilon \end{aligned}$$

(a) What does it mean to say that f is not ambivalent with respect to l at the point a.

$$\begin{aligned} \exists\epsilon &> 0, \exists\delta > 0, \forall\ x_1 \text{ and } \forall\ x_2 \\ |x_1 - a| &\geq \delta, \text{ or } |x_2 - a| \geq \delta, \text{ or } |f(x_1) - l| \geq \epsilon, \text{ or } |f(x_2) - l| \leq \epsilon \end{aligned}$$

(b) Find, if possible, an example of a function f that is ambivalent with respect to 2 at the point $x = 1$.

The condition $|f(x_1) - l| \geq \epsilon$ imples that f must take on values arbitrarily far from l, as well as taking on value arbitrarily close to l. So lets try

$$f(x) = \begin{cases} 2, & x = 1 \\ \dfrac{1}{x-1}, & x \neq 1 \end{cases}$$

Then for any $\epsilon > 0$ set $x_1 = 1$. Then we have

$$|x_1 - 1| = 0 < \delta \text{ and } |f(x_1) - l| = |2 - 2| = 0 < \epsilon$$

To see that we can find a value x_2 that satisfies the other two conditions, note that the expression $1/(x-1)$ can be made arbitrarily large by picking x close to 1. So for any ϵ and δ, which are positive pick x_2 so that

$$0 < x_2 - 1 < \min\{\delta, 1/\epsilon, 1\}.$$

For such a number we have

$$\begin{aligned} |x_2 - a| &= |x_2 - 1| = x_2 - 1 < \delta \text{ and} \\ |f(x_2) - l| &= \left|\frac{1}{x_2 - 1} - 2\right| \\ &= \left|\frac{3 - 2x_2}{x_2 - 1}\right| > \frac{|3 - 2x_2|}{1/\epsilon} \\ &= \epsilon\,|3 - 2x_2| > \epsilon \end{aligned}$$

Note, since $0 < x_2 - 1 < 1$, we have $|3 - 2x_2| > 1$.