1. (20) Define the following terms:

   (a) f is a function from the set A to the set B,

   A function from a set A to a set B is a subset of $A \times B$ such that

$$\begin{aligned} \forall a &\in A, \exists b \in B \text{ such that } (a, b) \in f \\ \forall a &\in A, (a, b_1) \text{ and } (a, b_2) \text{ in } f \text{ implies } b_1 = b_2 \end{aligned}$$

   (b) a one-to-one function,

   f is one-to-one if $f(x) = f(y)$ implies $x = y$.

   (c) an equivalence relation,

   An equivalence relation on a set A is a relation on A. That is, it is a subset of $A \times A$ such that it is reflexive, symmetric, and transitive.

   (d) equivalence class.

   If R is an equivalence relation on a set A, the equivalence class generated by a is,

$$[a] = \{x \in A : aRx\} .$$

2. (40) Let f be a function from the real numbers into the rational numbers, that is $f : R \to Q$, defined by

$$f(x) = \frac{\lfloor x \rfloor}{2} .$$

   where $\lfloor x \rfloor$ denotes the greatest integer function.

   (a) Is f onto?

   f is not onto. The equation $f(x) = \frac{1}{3}$ does not have a solution. If x satisfies $f(x) = 1/3$, then we have

$$\begin{aligned} \frac{\lfloor x \rfloor}{2} &= \frac{1}{3} \text{ or} \\ \lfloor x \rfloor &= \frac{2}{3} . \end{aligned}$$

   But $\lfloor x \rfloor$ is an integer and $2/3$ is not.

(b) Is f one-to-one?

f is not one-to-one. $f(1) = 1/2$ and $f(1.1) = 1/2$ also.

(c) In this part let f denote the map from the power set of R into the power set of Q, which is defined by the original f. If $X = [-1,\, 1] \cup (2\,,3)$, what does $f(X)$ equal? The notation $[a,\, b]$ denotes the set of real numbers between a and b with these two values also included. That is, $[a,\, b] = \{x : a \leq x \leq b\}$.

$$\begin{aligned} f(X) &= f([-1,\, 1]) \cup f((2\,,3)) \\ &= \left\{ \frac{-1}{2},\, 0,\, \frac{1}{2} \right\} \cup \{1\}\ . \end{aligned}$$

(d) In this part f^{-1} denotes the map from the power set of Q into the power set of R, which is defined by the original f. If $Y = \{-5,\, 2\}$, what does $f^{-1}(Y)$ equal?

$$\begin{aligned} f^{-1}(Y) &= f^{-1}(\{-5,\, 2\}) = f^{-1}(\{-5\}) \cup f^{-1}(\{2\}) \\ &= [-10,\ -9) \cup [4,\, 5) \end{aligned}$$

3. (30) Define a relation R on the set Z of integers by

$$mRn \iff \exists k \in Z \text{ such that } m - n = 4k\,.$$

(a) Show that R is an equivalence relation,

Since $m - m = 4 \cdot 0$, we have mRm.
If mRn, then there is a $k \in Z$ such that $m - n = 4k$, this implies $n - m = 4\,(-k)$. Since $-k \in Z$, we have nRm.
If mRn and nRp, there are integers k_1 and k_2 such that

$$\begin{aligned} m - n &= 4k_1 \\ n - p &= 4k_2. \end{aligned}$$

Adding these equations we have $m - p = 4\,(k_1 + k_2)$. Thus, mRp.

(b) Let $[m]$ denote the equivalence class generated by m. Show that $\{[0],\, [1],\, [2],\, [3]\}$ is a list of all the different equivalence classes.

Let m be any integer. Then the remainder when m is divided by 4 must be r, where r is one of 0, 1, 2, or 3. That is, $m = 4k + r$, or $m - r = 4k$ for some integer k. Thus, mRr, and we see that $\{[0],\, [1],\, [2],\, [3]\}$ lists all of the equivalence classes, and clearly they are different.

(c) Define addition of equivalence classes by

$$[m] + [n] = [m + n] .$$

Show that this is well defined, and compute the addition table.

Suppose $m_1 R m_2$ and $n_1 R n_2$. Then there are integers k_1 and k_2 such that

$$\begin{aligned} m_1 - m_2 &= 4k_1 \\ n_1 - n_2 &= 4k_2 \ . \end{aligned}$$

Adding we get

$$(m_1 + n_1) - (m_2 + n_2) = 4\,(k_1 + k_2) \ .$$

Thus, $(m_1 + n_1)\, R\, (m_2 + n_2)$. Hence addition of equivalence classes is well defined. The addition table for this binary operation on the equivalence classes is

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

4. (10) Let a_i and b_i denote arbitrary real numbers. The notation $\sum_{i=1}^{n} a_i$ is interpreted to mean $a_1 + a_2 + \cdots + a_n$. Use induction to prove

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \,,$$

for every natural number n.

The formula is certainly true for $n = 1$. Suppose it's true for n. Then

$$\begin{aligned} \sum_{i=1}^{n+1} (a_i + b_i) &= \sum_{i=1}^{n} (a_i + b_i) + (a_{n+1} + b_{n+1}) \\ &= \left(\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \right) + (a_{n+1} + b_{n+1}) \\ &= \left(\sum_{i=1}^{n} a_i + a_{n+1} \right) + \left(\sum_{i=1}^{n} b_i + b_{n+1} \right) \\ &= \sum_{i=1}^{n+1} a_i + \sum_{i=1}^{n+1} b_i \ . \end{aligned}$$

Thus, the formula is true for $n + 1$, and by induction the formula is valid for all natural numbers.