1. (10) Write a statement that is logically equivalent to the conditional statement $P \to Q$, using only "and ($\wedge$)", "or ($\vee$)" and "negation". Be sure to prove that the two statements are logically equivalent.

The following two statements are logically equivalent as the truth table below shows.

$$(P \to Q) \leftrightarrow (-P \vee Q)$$

P	Q	$P \to Q$	$-P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

2. (25) Let A, B, and C denote sets. Let $P(A)$ denote the power set of A.

a. Define $A \cup B$, and $A \cap B$.

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

b. True or false $C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$ for arbitrary sets A, B, and C. If true, prove it, if false give a counter example.

The equality is true. Suppose $x \in C \cap (A \cup B)$, then $x \in C$ and $x \in (A \cup B)$. If $x \in (A \cup B)$, then $x \in A$ or $x \in B$, and if $x \in A$, then $x \in C \cap A$. If $x \in B$, then $x \in C \cap B$. In either case $x \in (C \cap A) \cup (C \cap B)$. Thus the left hand side is a subset of the right hand side. Conversely if $x \in (C \cap A) \cup (C \cap B)$, then $x \in (C \cap A)$ or $x \in (C \cap B)$. In either case we have $x \in C$ and $x \in A \cup B$. Thus $x \in C \cap (A \cup B)$, and the right hand side is a subset of the left hand side, which means that the two sets are indeed equal.

c. True or false: if $A \sqsubseteq B$, then $P(A) \sqsubseteq P(B)$ for arbitrary sets A and B. If true, prove it, if false give a counter example.

It is true that $P(A) \sqsubseteq P(B)$. Suppose $x \in P(A)$. That means $x \sqsubseteq A$. Since A is a subset of B, then $x \sqsubseteq B$, or $x \in P(B)$.

3. (15) We say that the limit of $f(x)$ as x approaches a from above equals l, and write $\lim_{x \to a^+} f(x) = l$, if

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } \forall x \text{ if } 0 < x - a < \delta, \text{ then } |f(x) - l| < \epsilon \,.$$

a. Write the negation of this statement.

$$\exists \epsilon > 0, \forall \delta > 0, \text{ such that } \exists x \text{ with } 0 < x - a < \delta, \text{ and } |f(x) - l| \geq \epsilon$$

b. Write what it means to say that the limit of $f(x)$ as x approaches a from below equals l.

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } \forall x \text{ if } 0 < a - x < \delta, \text{ then } |f(x) - l| < \epsilon \,.$$

c. If $a = 5$ and $l = 2$, give an example of a function for which the limit of $f(x)$ as x approaches a from above does not equal l.

Let

$$f(x) = \begin{cases} 6, & x \geq 5 \\ 1, & x < 5 \end{cases} .$$

It is clear that $\lim_{x \to 5^+} f(x) = 6$ and hence is not equal to 2. To see that the limit is not 2, we verify that this function satisfies the conditions of the answer to part a. Set $\epsilon = 1$, actually any number less than 4 (the difference between 6 and 2) works. For any $\delta > 0$ set $x = 5 + \delta/2$. Then we have $0 < x - 5 = \delta/2 < \delta$, and

$$|f(x) - 2| = |f(5 + \delta/2) - 2| = |6 - 2| = 4 \geq 1 = \epsilon \,.$$

4. (20) Let N denote the natural numbers $\{1, 2, \cdots\}$.

a. State Peano's induction axiom for the natural numbers.

Any subset S of the natural numbers with the following two properties must equal N:

$$1 \in S$$

$$\text{if } n \in S, \text{ then } n + 1 \in S \,.$$

b. Prove, for every natural number $n \geq 7$, that

$$3^n < n! \,,$$

where $n!$ mean1 $1 \cdot 2 \cdot 3 \cdots \cdot n$. That is, $3! = 6$, $4! = 24$, etc.

We first note that the inequality is true for $n = 7$, since

$$3^7 = 2187 \text{ and } 7! = 5040 \,.$$

Thus, $3^7 < 7!$.

Assume now that $n \geq 7$ and $3^n < n!$. Then we have

$$\begin{aligned} 3^{n+1} &= 3 \cdot 3^n \\ &< 3 \cdot n! < (n+1)n! \\ &= (n+1)!. \end{aligned}$$

Thus, the inequality is true for all $n \geq 7$.

5. (15) Define what it means to say that the integer d divides the integer n. Then show: if $d|a$ and $d|b$, then $d|(ax + by)$ for any two integers x and y.

To say that d divides n, $d|n$, means there is an integer k such that

$$n = dk\,.$$

Suppose now that $d|a$ and $d|b$. Then there are integers k_1 and k_2 such that

$$a = k_1 d$$
$$b = k_2 d\,.$$

Thus,

$$a + b = (k_1 + k_2)d,$$

and we see that d does indeed divide $a + b$.

6. (15) Find all solutions of the Diophantine equation

$$15x + 21y = 36\,.$$

The greatest common divisor of 15 and 21 is 3, which divides 36. Thus, the equation has integer solutions. Since

$$3 = 3 \cdot 15 - 2 \cdot 21,$$

we have

$$\begin{aligned} 36 &= 12 \cdot 3 = 12 \cdot (3 \cdot 15 - 2 \cdot 21) \\ &= 36 \cdot 15 + (-24) \cdot 21. \end{aligned}$$

Thus one solution is $x_0 = 36$, and $y_0 = -24$. All other solutions are of the form

$$x = x_0 + k\left(\frac{21}{3}\right) = 36 + 7k$$
$$y = y_0 - k\left(\frac{15}{3}\right) = -24 - 5k\,.$$