1. (15) Define the following terms:

a. $f : X \to Y$,

This denotes a function from X to Y. That is a subset of $X \times Y$ such that the following two conditions are true:

for all $x \in X$ there is a $y \in Y$ such that $(x, y) \in f$

if (x, y_1) and (x, y_2) are in f then $y_1 = y_2$.

b. $a \equiv b(\mathrm{mod}\, m)$,

This means that m divides $a - b$.

c. the integers a and b are relatively prime.

Two integers are relatively prime if there greatest common divisor is 1.

2. (15) Let N denote the set of natural numbers. Define a relation R on N by

$$(m, n) \in R,$$

if $m|n$ or $n|m$. That is, the natural numbers m and n are related if one of them divides the other.

a. Determine if the relation is reflexive.

The relation is reflexive, for if $m \in N$, then m divides m, so $(m, m) \in R$.

b. Determine if the relation is symmetric.

The relation is symmetric, for if $(m, n) \in R$, then either m divides n or n divides m, and this is the same as saying either n divides m or m divides n, or $(n, m) \in R$.

c. Determine if the relation is transitive.

The relation is not transitive as the following example shows.

$(8, 4) \in R$ and $(4, 12) \in R$, but $(8, 12)$ is not in R.

3. (10) Let $f : A \to B$.

a. Explain how f defines a function, denoted by f again, from $P(A)$, the power set of A to $P(B)$ the power set of B.

If $S \in P(A)$, then $f(S) = \{f(a) : a \in S\} \subseteq B$. That is, $f(S) \in P(B)$.

b. Explain how f defines a function, denoted by f^{-1}, from $P(B)$ to $P(A)$.

If $Q \in P(B)$, then $f^{-1}(Q) = \{a \in A : f(a) \in Q\} \subseteq A$. So $f^{-1}(Q) \in P(A)$.

4. (10) We saw that it is possible to express any natural number in a base besides 10. If $N = 53,024$ in base 10, and we write it in base 3 we have

$$N = b_n 3^n + b_{n-1} 3^{n-1} + \cdots + b_1 3 + b_0 ,$$

where each of the b_i's are integers between 0 and 2. For the given N determine b_1 and b_0.

If $N = N = b_n 3^n + b_{n-1} 3^{n-1} + \cdots + b_1 3 + b_0$, $N - (b_1 3 + b_0)$ must be divisible by 9. That is, N is congruent to $b_1 3 + b_0 \bmod 9$. Dividing N by 9 we get

$$53,024 = 5,891 \times 9 + 5 .$$

Hence $b_1 3 + b_0 = 5$. And 5 in base three is 12. So $b_1 = 1$ and $b_0 = 2$.

5. (13) Let $f : X \to Y$ and $g : Y \to X$ be two functions with the property that

$$(f \circ g)(y) = y ,$$

for every $y \in Y$. Show that g must be one-to-one.

Suppose y_1 and y_2 are such that $g(y_1) = g(y_2)$. This implies

$$y_1 = f(g(y_1)) = f(g(y_2)) = y_2 .$$

Thus, g is a one-to-one function.

6. (12) Let a and n be relatively prime integers with $n > 1$. Show there is an integer b such that

$$ab \equiv 1 (\bmod n) .$$

Since a and n are relatively prime (their greatest common divisor is 1), there are integers x and y such that

$$1 = ax + ny .$$

That is, $1 - ax$ is divisible by n, or

$$ax \equiv 1 (\bmod n) ,$$

and we have found our b.