Feb. 20, 1997 Math 220 Exam 1

A yes or no answer or an answer with no justification will not be acceptable. Remember to write neatly, clearly, and in sentences.

1. (20) Define the following terms and/or symbols, and give an example of each. No example, no credit.

   (a) contrapositive

   Ans: If we have an implication $P \to Q$, its contrapositive is the implication $\neg Q \to \neg P$. As an example, the contrapositive of the statement, 'If 2 is less than 3, then 3 is less than 4', is the statement, 'if 3 is greater than or equal to 4 then 2 is greater than or equal to 3'.

   (b) tautology

   Ans: A tautology is a statement that is always true regardless of the truth values of its component statements. An example of a tautology is $P \vee \neg P$.

   (c) $A - B$, where A and B are two sets

   Ans: $A - B$ is defined as the set of elements that are in A and not in B. If we let $A = \{1, 2, 3\}$ and $B = \{1, 2\}$, then $A - B = \{3\}$.

   (d) $\bigcup_{\gamma \in \Gamma} A_\gamma$

   Ans: In this notation , is referred to as an indexing set. That is, each member γ of , has a set associated with it, A_γ, and the notation $\bigcup_{\gamma \in \Gamma} A_\gamma$ represents the set which consists of all elements that are in at least one of the sets A_γ. For an example, let , $= \{1, 2, 3\}$, $A_1 = \{2\}$, $A_2 = \{3\}$, and $A_3 = \{4\}$. Then, $\bigcup_{\gamma \in \Gamma} A_\gamma = \{2, 3, 4\}$.

2. (20) Is the following a tautology, a contradiction, or neither?

$$[(P \to Q) \wedge (Q \to P)] \to (P \wedge Q) \vee (\neg P \wedge \neg Q)$$

   Ans: This is a tautology. One way to see this is to construct a truth table for this statement and verify that it is always true.

3. (20) Prove or disprove the following:

$$(A - B) - C = A - (B - C),$$

where A, B, and C are sets.

Ans: This is not a true statement. To see how to construct a counter example notice that the LHS can contain no elements in the set C, while that is not true for the RHS. Thus, let $A = \{1\}$, $B = \{2\}$, and $C = \{1\}$. Then $(A - B) - C = \emptyset$, and $A - (B - C) = \{1\}$.

---

4. (15) Assign a grade of A (correct), C (partially correct), F (wrong) to the following "proof". Justify all grades other than an A.

Claim: Let $\mathcal{N}$ denote the natural numbers. If $X = \{x \in \mathcal{N} : x^2 < 14\}$ and $Y = \{1, 2, 3\}$, then $X = Y$.

Proof: Since $1^2 = 1 < 14$, $2^2 = 4 < 14$, and $3^2 = 9 < 14$, we have $X = Y$.

Ans: I would give a grade of C. This proof is not complete. It does correctly show that the set Y is a subset of the set X. However, there is no indication that the author of this 'proof' ever worried about whether or not X is a subset of Y.

---

5. (15) Let $A = \{1, 2, 3, 4\}$. Let $\mathcal{P}(A)$ denote the power set of A. Which, if any, of the following are true statements? Be sure to explain.

(a) $1 \in \mathcal{P}(A)$

Ans: This is not true. The element 1 is a member of A, but it is not a subset of A. Since $\mathcal{P}(A)$ is defined to be the collection of all subsets of A, 1 does not belong to $\mathcal{P}(A)$.

(b) $\{1, 2\} \subseteq \mathcal{P}(A)$

Ans: This is a false statement. Each of the elements 1 and 2 belong to A. Thus, $\{1, 2\}$ is a subset of A and belongs to $\mathcal{P}(A)$. Thus, we should write $\{1, 2\} \in \mathcal{P}(A)$. When we write $\{1, 2\} \subseteq \mathcal{P}(A)$, we are saying that $\{1, 2\}$ is a subset of $\mathcal{P}(A)$. This means we are saying that both of the elements 1 and 2 belong to $\mathcal{P}(A)$, and as we saw in part (a), this is not true.

(c) $A \in \mathcal{P}(A)$.

Ans: This is a true statement. Clearly every element of A belongs to A. Thus, A is a subset of itself, and A is a member of $\mathcal{P}(A)$.

---

6. (10) For each $\alpha \in \mathcal{A}$ let A_α be a set. In class we saw that the following set inclusion is true:

$$\bigcup_{\alpha,\beta \in \mathcal{A}} (A_\alpha - A_\beta) \subseteq \bigcup_{\alpha \in \mathcal{A}} A_\alpha - \bigcap_{\alpha \in \mathcal{A}} A_\alpha .$$

Is it possible for the left hand side to be a proper subset of the right hand side or must the two sides always equal each other?

Ans: It is not possible for the LHS to be a proper subset of the RHS. The two sets are equal. To see this suppose x belongs to the RHS. Then there are α_0 and β_0 such that $x \in A_{\alpha_0}$ and $x \notin A_{\beta_0}$. Thus, we have $x \in A_{\alpha_0} - A_{\beta_0}$. Hence x belongs to the LHS also, and we have shown the RHS is a subset of the left hand side. Since we already know that the LHS is a subset of the RHS, the two sets are equal.
