

A yes or no answer or an answer with no justification will not be acceptable. Remember to write neatly, clearly, and in **sentences**.

1. (15) Define the following terms, and give an example of each.

(a) Relation in $A \times B$,

Ans: A relation in $A \times B$ is a subset of $A \times B$. As an example let $A = \{1, 2\}$ and $B = \{\alpha\}$. Let $S = \{(2, \alpha)\}$, then S is a relation in $A \times B$.

(b) If S is a relation in $A \times B$, what is $S[x]$?

Ans: $S[x] = \{y \in B : (x, y) \in S\}$. For the relation S in part (a), $S[1] = \emptyset$, and $S[2] = \{\alpha\}$.

(c) The domain of a relation S in $A \times B$.

Ans: The domain of $S = \{x \in A : \exists y \in B \text{ such that } (x, y) \in S\}$. For our S , its domain is $\{2\}$.

2. (15) Let $a_1 = -1$, $a_2 = 1$, and $a_3 = 3$. Suppose that for each $n > 3$ we have $a_n = a_{n-1} + a_{n-2} - a_{n-3}$. Find a formula for a_n and prove your conjecture.

Ans: The first thing to do is to compute some values of a_n and attempt to discern a pattern.

$n :$	1	2	3	4	5	6	7	8	9	10
$a_n :$	-1	1	3	5	7	9	11	13	15	17

Some trial and error soon lead to the conjecture that $a_n = 2n - 3$, and this formula is checked to see that it is valid for each of the a_n in the table. Thus, to verify the validity of this formula for all n we only need to check the inductive step. So let us assume the formula is true for $1 \leq k \leq n$ and verify that it is then true for $n + 1$.

$$a_{n+1} = a_n + a_{n-1} - a_{n-2} = (2n-3) + (2(n-1)-3) - (2(n-2)-3) = \dots = 2(n+1)-3.$$

Thus, by the second principle of induction our conjecture is seen to be correct.

3. (15) Let $f_1 = 1$, $f_2 = 1$, and $f_{n+2} = f_n + f_{n+1}$, for $n = 1, 2, \dots$. This sequence of numbers is called the Fibonacci sequence. Prove by induction that $\sum_{i=1}^{n-1} f_i = f_{n+1} - 1$ for each natural number $n > 1$. Be sure to state which principle of induction you are using.

Ans: Let $S = \left\{ n \in \mathbb{N} : \sum_{i=1}^{n-1} f_i = f_{n+1} - 1 \right\}$. One quickly checks that $1 \in S$. For the inductive step we assume that $n \in S$ and then

$$\sum_{i=1}^n f_i = \sum_{i=1}^{n-1} f_i + f_n = f_{n+1} - 1 + f_n = f_{n+2} - 1.$$

Thus, by the first principle of induction we have shown that $S = \mathbb{N}$.

4. (15) Let \mathcal{A} be a set and for each $\alpha \in \mathcal{A}$ let A_α be a nonempty set. Let B be any other nonempty set. Is the following a true statement:

$$\left(\bigcap_{\alpha \in \mathcal{A}} A_\alpha \right) \times B = \bigcap_{\alpha \in \mathcal{A}} (A_\alpha \times B),$$

where \times denotes the Cartesian product.

Ans: Let $x = (a, b)$ be an element of the LHS. Then $a \in \bigcap_{\alpha \in \mathcal{A}} A_\alpha$ and $b \in B$. Thus, for each $\alpha \in \mathcal{A}$, $a \in A_\alpha$ from which we conclude that $x = (a, b) \in A_\alpha \times B$ for each $\alpha \in \mathcal{A}$. Thus, x is also an element of the RHS, and the LHS is a subset of the RHS. If x is an element of the RHS, then $x = (a, b) \in A_\alpha \times B$ for each $\alpha \in \mathcal{A}$. Thus, for each $\alpha \in \mathcal{A}$, $a \in A_\alpha$, and we have $a \in \bigcap_{\alpha \in \mathcal{A}} A_\alpha$. Hence $x = (a, b)$ is also an element of the LHS, and the two sets are equal.

5. (15) Use the Euclidean algorithm to find the greatest common divisor of 15,435 and 2,145. You do not have to write $\gcd(15435, 2145)$ as a linear combination of these two numbers.

Ans:

$$15435 = 2145(7) + 420$$

$$2145 = 420(5) + 45$$

$$420 = 45(9) + 15$$

$$45 = 15(3).$$

Thus, the greatest common divisor of these two integers is 15.

6. (10) Work either of the following two problems.

- (a) The least common multiple, $\text{lcm}(a, b)$, of two natural numbers, a and b , is defined to be the least natural number which is divisible by both a and b . Is it true that for any natural numbers a and b there exist integers x and y such that $\text{lcm}(a, b) = ax + by$?

Ans: The easiest solution I saw is the following: let l be the least common multiple of a and b . Then there is an integer, k , such that $l = ka$. Thus, $l = a(k) + b(0)$, and we have written the least common multiple of a and b as an integer linear combination of them.

- (b) Let the natural numbers a and c be relatively prime. Suppose that c divides the product ab . Show that c divides b .

Ans: Since a and c are relatively prime, i.e., the greatest common divisor is 1, there exist integers x and y such that $1 = ax + cy$. Multiplying this equation by b we have

$$b = abx + bcy = c(kx) + bcy = c(kx + by)$$

where $ab = kc$ follows from the fact that c divides ab . Thus, we see that c divides b .

7. (15) Consider the Diophantine equation $30x + 12y = 192$.

(a) Find one solution of this equation.

Ans: Since the greatest common divisor of 30 and 12 is 6, and 6 divides 192, we know that the equation does have a solution. One of which follows from the equation

$$\begin{aligned} 192 &= 6(32) = (30 + 12(-2))32 \\ &= 30(32) + 12(-64). \end{aligned}$$

Thus the pair $x_0 = 32$ and $y_0 = -64$ is a solution to this Diophantine equation.

(b) If you label the solution pair you found in part (a) as x_0 and y_0 , show that any other solution has the form

$$x = x_0 + \frac{12}{d}n \quad \text{and} \quad y = y_0 - \frac{30}{d}n,$$

where d is the greatest common divisor of 30 and 12, and n is any integer.

Ans: If x and y are another solution pair, then the integers $z_1 = x - x_0$ and $z_2 = y - y_0$ satisfy the equation $30z_1 + 12z_2 = 0$. Divide this equation by 6 which is the greatest common divisor of 12 and 30. This leads to the equation:

$$5z_1 + 2z_2 = 0.$$

From which we may infer that 5 divides $2z_2$. Since 5 and 2 are relatively prime we know that 5 must also divide z_2 . Set $n = \frac{z_2}{5}$. Then we have $z_1 = -2(z_2/5) = -2n$. Thus we have

$$\begin{aligned} x &= x_0 + z_1 = x_0 + 2(-n) = x_0 + \frac{12}{6}(-n) \\ y &= y_0 + z_2 = y_0 + 5\frac{z_2}{5} = y_0 + \frac{30}{6}n \end{aligned}$$

Thus, every solution of this Diophantine equation has been shown to have the desired form.
