A yes or no answer or an answer with no justification will not be acceptable. Remember to write neatly, clearly, and in **sentences**.

1. (15) If n is an even integer, show that $\gcd(n, n+2) = 2$. What can you say about the $\gcd(n, n+2)$, if n is an odd integer?

Ans: Let d be the greatest common divisor of n and $n+2$. Then d must divide 2, since $2 = (n+2) - n$. Thus, d equals 1 or 2. Since n is even we know that 2 divides both n and $n+2$. Hence in the case where n is even $d = 2$. If n is odd, then d cannot equal 2, and in this case $d = 1$.

---

2. (10) We have defined congruence modulo n as follows: $a \equiv b \pmod{n}$ if and only if n divides $b - a$, where n is a natural number, and a and b are integers. Show that this is equivalent to the statement that a and b have the same remainder when divided by the natural number n.

Ans: Use the division algorithm to write $a = nq_1 + r_1$ and $b = nq_2 + r_2$, where $0 \le r_i < n$. Then $a - b = n(q_1 - q_2) + (r_1 - r_2)$. Suppose a and b have the same remainder. That is, $r_1 = r_2$. Then we have $a - b = n(q_1 - q_2)$, and a is congruent to b modulo n. Conversely suppose that a and b are congruent to each other modulo n. Then from the above equation, we have that n must divide $r_1 - r_2$. However, since both r_1 and r_2 satisfy the inequality $0 \le r_i < n$, their difference must lie strictly between $-n$ and n. The only integer satisfying this inequality and divisible by n is zero. Thus, a and b have the same remainders when divided by n.

---

3. (10) Show that 30 divides $n^{13} - n$ for every natural number n.

Ans: The key to this is to use Fermat's Little Theorem which states that if p is a prime number and a is not divisible by p, then $a^{p-1} \equiv 1 \bmod p$. The prime factors of 30 are 2, 3, and 5. What we want to do is to show than $n(n^{12} - 1) \equiv 0 \bmod n$. Where n is 2, 3, or 5. If n is divisible by 2, 3, or 5 the above congruence is certainly true. Thus, in each of following lines we assume that n is not divisible by the respective prime number.

$$
\begin{aligned}
n(n^{12} - 1) &\equiv n((n^1)^{12} - 1) \equiv n(1^{12} - 1) \equiv n \cdot 0 \equiv 0 (\bmod 2) \\
n(n^{12} - 1) &\equiv n((n^2)^6 - 1) \equiv n(1^6 - 1) \equiv n \cdot 0 \equiv 0 (\bmod 3) \\
n(n^{12} - 1) &\equiv n((n^4)^3 - 1) \equiv n(1^3 - 1) \equiv n \cdot 0 \equiv 0 (\bmod 5)
\end{aligned}
$$

Since these three primes are all relatively prime we have that $n^{13} \equiv n(\bmod (2 \cdot 3 \cdot 5))$.

---

4. (10) The professor tells Mary that it is necessary for her to get at least a C on the final in order for her to pass the course. Mary gets a C. What can she conclude?

   (a) She passed the course.

   (b) She can conclude nothing.

Ans: Mary can conclude nothing. If she had not received at least a C, she could have concluded that she did not pass the course. However, since making a C or better was necessary (not sufficient) she cannot conclude that she passed. In the language of logic the statement P is necessary for Q can be written $\neg P \to \neg Q$, or $Q \to P$.

---

5. (10) Assume that A and B are sets, and that P and Q are statements. Which of the following make sense mathematically, and which do not.

   (a) $B \subset A$,

Ans: Since A and B are sets the mathematical statement that A is a subset of B certainly makes sense.

(b) $P \cup Q$,

Ans: The set theoretic union of two logical statements does not make sense.

(c) $\forall x \in A,\ x \in P$.

Ans: This statement does not make sense. To talk about something belonging to a statement $(x \in P)$ is nonsensical.

---

6. (15) Let f_n denote the Fibonacci numbers. That is $f_1 = 1$, $f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for each natural number n. Show that $\displaystyle\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$ for each natural number n.

Ans: We first observe that the conjecture is true for $n = 1$. The heart of the problem is the inductive step which follows below.

$$\begin{aligned}\sum_{i=1}^{n+1} f_i^2 &= \sum_{i=1}^{n} f_i^2 + f_{n+1}^2 = f_n f_{n+1} + f_{n+1}^2 \\ &= f_{n+1}(f_n + f_{n+1}) = f_{n+1} f_{n+2}\,.\end{aligned}$$

---

7. (15) For each natural number i, let a_i be a real number. For each natural number n, define $\displaystyle\sum_{i=1}^{n} a_i$ as follows:

$$\sum_{i=1}^{1} a_i = a_1, \qquad \sum_{i=1}^{n+1} a_i = a_{n+1} + \sum_{i=1}^{n} a_i\,.$$

Show that $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$.

Ans: One easily verifies the statement is true for $n = 1$. The inductive argument is given below.

$$\begin{aligned}\sum_{i=1}^{n+1}(a_i + b_i) &= \sum_{i=1}^{n}(a_i + b_i) + (a_{n+1} + b_{n+1}) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i + (a_{n+1} + b_{n+1}) \\ &= \left[\sum_{i=1}^{n} a_i + a_{n+1}\right] + \left[\sum_{i=1}^{n} b_i + b_{n+1}\right] = \sum_{i=1}^{n+1} a_i + \sum_{i=1}^{n+1} b_i\,.\end{aligned}$$

---

8. (15) Let $\mathbb{R}^2$ denote the set of all ordered pairs of real numbers. Define the following relation S on $\mathbb{R}^2 \times \mathbb{R}^2$ by: (x, y) is related to (u, v) if $v - y = 2(u - x)$.

   (a) Show that this is an equivalence relation.

Ans: We need to verify that the relation is reflexive, symmetric, and transitive. The see that (x, y) is related to itself for every pair of real numbers, we need to verify that $y - y = 2(x - x)$, which is indeed true. This shows that the relation is reflexive. For symmetry suppose that (x, y) is related to (u, v). Then $v - y = 2(u - x)$. Multiplying this equation by minus one we have, $y - v = 2(x - u)$. Which shows that (u, v) is related to (x, y). In other words the relation is symmetric. Suppose finally that (x, y) is related to (u, v) and that this is related to (r, s). The following equations are then valid:

$$\begin{aligned} v - y &= 2(u - x) \\ s - v &= 2(r - u)\,. \end{aligned}$$

Adding the two equations, we get: $s - y = 2(r - x)$. That is, (x, y) is related to (r, s). All three properties of an equivalence relation have been verified and we have thereby shown that this relation is an equivalence relation.

   (b) Describe the equivalence class $S[(1, 2)]$ geometrically.

Ans: (x, y) is in the equivalence class $S[(1, 2)]$ if and only if (x, y) is related to $(1, 2)$, or if and only if $2 - y = 2(1 - x)$. Solving for y we get $y = 2x$. Thus, this equivalence class can be viewed as the straight line in $\mathbb{R}^2$ with slope two which passes through the origin.
