1. (20) Define the following

(a) $\lim_{(x,y)\to(2,5)} f(x,y) = 2.$

This means that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\text{if } 0 < \sqrt{(x-2)^2 + (y-5)^2} < \delta, \text{ then } |f(x) - 2| < \epsilon.$$

(b) $f(x,y)$ is differentiable at the point $(-1,2)$.

To say f is differentiable at the point $(-1,2)$ means that the first partials of f exist at that point and there are numbers ϵ_1 and ϵ_2 such that

$$f(-1+\Delta x, 2+\Delta y) = f(-1,2) + f_x(-1,2)\Delta x + f_y(-1,2)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $\lim_{(\Delta x,\Delta y)\to(0,0)} \epsilon_i = 0$ for $i = 1$ or 2.

(c) The directional derivative of $f(x,y,z)$ at the point $(1,2,3)$ in the direction $\vec{N} = \left(-2, 4, \sqrt{5}\right)$.

$D_N f(1,2,3)$ is defined as

$$D_{\vec{N}}\ f(1,2,3) = \lim_{h\to 0} \frac{f\left(1 + h(-2/5), 2 + h(4/5), 3 + h(\sqrt{5}/5\right) - f(1,2,3)}{h}.$$

Note: The vector $\vec{N}$ was divided by its length, 5, to get a unit vector in the same direction as $\vec{N}$.

(d) $f(x,y)$ has a local minimum at the point $(-1,2)$

To say that f has a local minimum ath the point $(-1,2)$ means that there is some rectangle, R, that contains the point $(-1,2)$ in its interior and for any point $(x,y) \in R$ we have

$$f(x,y) \geq f(-1,2).$$

2. (10) The vector $\vec{X} = (3,2,0)$ represents a force on a particle. Find two vectors $\vec{A}$ and $\vec{B}$ such that $\vec{X} = \vec{A} + \vec{B}$, where $\vec{A}$ is parallel to the vector $\vec{N} = (1,1,1)$, and $\vec{B}$ is perpendicular to $\vec{N}$.

The vector $\vec{A}$ is the projection of $\vec{X}$ onto $\vec{N}$. Thus,

$$\begin{aligned} \vec{A} &= \frac{\vec{X}\cdot\vec{N}}{\left\|\vec{N}\right\|^2}\vec{N} = \frac{5}{3}(1,1,1) \\ \vec{B} &= \vec{X} - \vec{A} = (3,2,0) - \frac{5}{3}(1,1,1) = \left(\frac{4}{3}, \frac{1}{3}, -\frac{5}{3}\right). \end{aligned}$$

Clearly $\vec{X} = \vec{A} + \vec{B}$. Moreover, since $\vec{A}$ is a scalar multiple of $\vec{N}$, it is parallel to $\vec{N}$, and a quick check shows that $\vec{B}$ is perpendicular to $\vec{N}$.

3. (20) Let $P=(1,1,0)$, $Q=(-1,0,3)$, $R=(2,-1,1)$.

(a) Find an equation for the plane passing through the above three points.

The two vectors $Q-P=(-2,-1,3)$ and $R-P=(1,-2,1)$ lie in the desired plane. Hence, their cross product $\vec{N}=5\,(1,1,1)$, is perpendicular to the plane. If the point (x,y,z) lies on the plane then we must have

$$\begin{aligned} (x-1,y-1,z-0)\cdot(1,1,1) &= 0 \\ &\text{or} \\ x+y+z &= 2 \end{aligned}$$

(b) Find an equation for the straight line passing through the point P and the point $(1,-1,1)$.

A vector parallel to this line is $(1,-1,1)-P=(0,-2,1)$. Thus, an equation for the line is

$$\begin{aligned} \Gamma(t) &= (1,1,0)+t\,(0,-2,1) \\ &\text{or} \\ x &= 1,\ y=1-2t,\ z=t \end{aligned}$$

(c) What is the distance from the point $(1,-1,1)$ to the plane you found in part a?

The distance will equal the length of the projection of a vector from any point on the plane to the point $(1,-1,1)$ onto a direction perpendicular to the plane. Such a vector is $(0,-2,1)$ and $(1,1,1)$ is perpendicular to the plane. Thus

$$\begin{aligned} \text{distance} &= \left|\left|\text{Proj}_{(1,1,1)}(0,-2,1)\right|\right| = \left|\left|\frac{(1,1,1)\cdot(0,-2,1)}{3}(1,1,1)\right|\right| \\ &= \left|\frac{-1}{3}\right|\,||(1,1,1)|| = \frac{1}{3}\sqrt{3} = \frac{1}{\sqrt{3}} \end{aligned}$$

4. (20) Let $f(x,y,z) = x^2 - 3xy^2 + xyz$.

We first calculate the values of the three first partial derivatives of f at the point $(1,1,1)$. They are

$$\begin{aligned} f_x &= \left(2x - 3y^2 + yz\right)\big|_{(1,1,1)} = 0 \\ f_y &= (-6xy + xz)|_{(1,1,1)} = -5 \\ f_z &= (xy)|_{(1,1,1)} = 1 \end{aligned}$$

(a) Compute the rate of change of f at the point $(1,1,1)$ in the direction given by $\vec{N} = (-1,2,3)$.

$$\begin{aligned} D_{\vec{N}}\, f(1,1,1) &= \nabla f(1,1,1) \cdot \frac{(-1,2,3)}{\sqrt{14}} \\ &= 0\frac{-1}{\sqrt{14}} + (-5)\frac{2}{\sqrt{14}} + (1)\frac{3}{\sqrt{14}} \\ &= \frac{-7}{\sqrt{14}} \end{aligned}$$

(b) Using the differential of f at the point $(1,1,1)$ approximate $f(0.9, 1.01, 0.95)$.

$$\begin{aligned} f(0.9,1.01,0.95) &\approx f(1,1,1) + f_x|_{(1,1,1)}(-0.1) + f_y|_{(1,1,1)}(0.01) + f_z|_{(1,1,1)}(-0.05) \\ &= -1 + 0 - 5(0.01) - 0.05 \\ &= -1.1. \end{aligned}$$

(c) Find an equation for the tangent plane to the surface $f = -1$ at the point $(1,1,1)$.

A normal to the tangent plane is given by the gradient of f evaluated at the point $(1,1,1)$. Thus, an equation for this tangent plane is

$$\begin{aligned} (x-1, y-1, z-1) \cdot (0,-5,1) &= 0 \\ &\text{or} \\ 5y - z &= 4 \end{aligned}$$

(d) Let $\Gamma(t) = (x(t), y(t), z(t))$ be a curve which passes through the point $(1,1,1)$ when $t = 1/2$. Suppose $\Gamma'(1/2) = (3,-2,11)$. Set $g(t) = f(\Gamma(t))$. What must $g'(1/2)$ equal?

$$\begin{aligned} g'(1/2) &= \nabla f|_{(1,1,1)} \cdot \Gamma'(1/2) \\ &= (0,-5,1) \cdot (3,-2,11) \\ &= 21 \end{aligned}$$

5. (10) Let $f(x,y) = \dfrac{xy}{x^2+y^2}$.

(a) Find $\lim\limits_{(x,y)\to(1,2)} f(x,y)$

$$\lim_{(x,y)\to(1,2)} \frac{xy}{x^2+y^2} = \frac{1\cdot 2}{1^2+2^2} = \frac{2}{5}$$

(b) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

This limit does not exist. If we let $(x,y)\to(0,0)$ along the line $y=0$, a limiting value of 0 occurs. However, if we let $(x,y)\to(0,0)$ along the line $y=x$ a limiting value of $1/2$ occurs. Thus, no limit as $(x,y)\to(0,0)$ can exist.

6. (20) Let $f(x,y) = x^2+y^2+x^2y+4$

(a) What is the domain of this function?

The domain is all of R^2.

(b) Find all critical points. That is, find all possible locations of local extrema.

$$\nabla f = \left(2x+2xy, 2y+x^2\right) = (0,0)$$

implies

$$\begin{aligned} 2x(1+y) &= 0 \\ 2y+x^2 &= 0. \end{aligned}$$

The first equation implies either $x=0$ or $y=-1$. If $x=0$, then the second equation implies $y=0$. If $y=-1$, then the second equation implies $x=\pm\sqrt{2}$. So the critical points are

$$(0,0)\,,\ \left(\sqrt{2},-1\right),\ \text{and}\ \left(-\sqrt{2},-1\right)$$

(c) At each of the points you found in part b. determine what sort of singular point it is.

The table below contains the various values of f_{xx} and $f_{xx}f_{yy}-(f_{xy})^2$, and we have

$$f_{xx} = 2+2y,\ f_{yy} = 2,\ f_{xy} = 2x$$

(x,y)	f_{xx}	f_{yy}	$f_{xx}f_{yy}-(f_{xy})^2$	
$(0,0)$	2	2	4	local min
$\left(\sqrt{2},-1\right)$	0	2	-8	saddle point
$\left(-\sqrt{2},-1\right)$	0	2	-8	saddle point