1. (15) Define the following:

(a) $\lim_{(x,y)\to(-2,1)} f(x,y) = 5$

This means that for any $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < \sqrt{(x+2)^2 + (y-1)^2} < \delta$, then $|f(x,y) - 5| < \epsilon$.

(b) State the mean value theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable in the open interval (a, b). Then there is a point $\xi \in (a, b)$ such that

$$f(b) - f(a) = f'(\xi)(b - a)$$

(c) State Green's theorem.

Let C be a closed piece wise smooth curve and let int(C) denote the region contained inside C. Then for any two functions P and Q that are continuously differentiable on C and its interior, we have

$$\oint_C P dx + Q dy = \int\int_{\text{int}(C)} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA.$$

where the direction of integration around the curve C is such that the interior of C is to the left when one walks around the path C in the direction of integration.

2. (20) P, Q, R, and ϕ are real valued functions with domains equal to R^3, each of which is twice continuously differentiable. Let $\vec{F} = (P, Q, R)$.

(a) Show that $\text{div}\left(\text{curl}\left(\vec{F}\right)\right) = 0$.

$$\begin{aligned}
\text{div}\left(\text{curl}\left(\vec{F}\right)\right) &= \text{div}\left(\text{curl}\left(P, Q, R\right)\right) \\
&= \text{div}\left(R_y - Q_z, P_z - R_x, Q_x - P_y\right) \\
&= \frac{\partial}{\partial x}\left(R_y - Q_z\right) + \frac{\partial}{\partial y}\left(P_z - R_x\right) + \frac{\partial}{\partial z}\left(Q_x - P_y\right) \\
&= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} \\
&= \left(R_{yx} - R_{xy}\right) + \left(Q_{xz} - Q_{zx}\right) + \left(P_{zy} - P_{yz}\right) \\
&= 0
\end{aligned}$$

(b) Show that $\text{curl}(\text{grad}(\phi)) = \vec{0}$.

$$\begin{aligned}
\text{curl}\left(\text{grad}\left(\phi\right)\right) &= \text{curl}\left(\phi_x, \phi_y, \phi_z\right) \\
&= \left(\phi_{zy} - \phi_{yz}, \phi_{xz} - \phi zx, \phi_{yx} - \phi_{xy}\right) \\
&= (0, 0, 0) \ .
\end{aligned}$$

3. (20) Let D equal that part of the unit disk centered at the origin that lies to the right of the y axis. Let $\vec{F}(x,y) = (x,y)$.

(a) Compute the line integral of the tangential component of $\vec{F}$ around the boundary of D in a counter clockwise direction.

Let C denote the boundary of D oriented in the standard counter clockwise direction. Then we have

$$\oint_C x dx + y dy = \iint_D \left[\frac{\partial(y)}{\partial x} - \frac{\partial(x)}{\partial y} \right] dA = \iint_D 0 \; dA = 0$$

You can also observe that $\vec{F}$ is a conservative force field and C is a closed path, to see that the line integral must be zero

(b) Compute the line integral of the normal component of $\vec{F}$ around the boundary of D in a counter clockwise direction.

Let C denote the boundary of D oriented in the standard counter clockwise direction. Let $(x(t), y(t))$ denote a parametrization of C, then (x', y') is tangent to the curve. Thus, the outward normal direction is given by $(y', -x')$, if the curve is traversed in the counter clockwise direction. Then we have

$$\begin{aligned}
\oint_C (x,y) \cdot \frac{(y', -x')}{\sqrt{(x')^2 + (y')^2}} ds &= \oint_C (x,y) \cdot (dy, -dx) \\
&= \oint_C -y dx + x dy = \iint_D \left[\frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y} \right] dA \\
&= \iint_D 2 \; dA = 2 \text{area}(D) \\
&= 2\frac{\pi}{2} = \pi \; .
\end{aligned}$$

4. (10) Let $\vec{F}(x,y,z) = \big(x^2+2z,\ y-x,\ xy\big)$. Let C denote the path that goes from $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$ along straight line segments. That is, $C : (1,0,0) \to (1,1,0) \to (1,1,1)$ and each arrow represents the straight line joining the two points. Compute the tangential component of $\vec{F}$ along the path C.

Let C_1 and C_2 denote the two straight line paths that make up C. So on C_1 we have $x=1$ and $z=0$, while on C_2 we have $x=1$ and $y=1$. Thus,

$$\begin{aligned}
\int_C F\cdot dr &= \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr \\
&= \int_0^1 (1, y-1, y)\cdot(0, dy, 0) + \int_0^1 (1+2z, 0, 1)\cdot(0,0,dz) \\
&= \int_0^1 (y-1)\ dy + \int_0^1 dz = \left(\frac{y^2}{2} - y\right)\Bigg|_0^1 + 1 \\
&= \left(\frac{1}{2} - 1\right) - 0 + 1 = \frac{1}{2}
\end{aligned}$$

5. (20) Let $\vec{F}(x,y,z) = \left(y^2,\, 2xy+z^2,\, 2yz+3z^2\right)$.

(a) Find a function ϕ such that $\nabla\phi = \vec{F}$.
Such a ϕ must satisfy the partial differential equations

$$\frac{\partial \phi}{\partial x} = y^2,\ \frac{\partial \phi}{\partial y} = 2xy+z^2,\ \frac{\partial \phi}{\partial z} = 2yz+3z^2\ .$$

If the last equation is true, then $\phi = yz^2 + z^3 + c(x,y)$ for some unknown function c of x and y. The second equation tells us that ϕ must satisfy the equation

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= 2xy + z^2 = z^2 + \frac{\partial c}{\partial y} \text{ or} \\ \frac{\partial c}{\partial y} &= 2xy \text{ or} \\ c &= xy^2 + d(x)\ . \end{aligned}$$

Thus, if ϕ must equal $\phi = yz^2 + z^3 + xy^2 + d(x)$. Then the first of the original equations forces the following to be true

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= y^2 = y^2 + d'(x) \text{ or} \\ d'(x) &= 0 \text{ or} \\ d &= \text{constant}\ . \end{aligned}$$

Thus, for any constant k, the function ϕ below is a potential function for the given force field.

$$\phi(x,y,z) = yz^2 + z^3 + xy^2 + k\ .$$

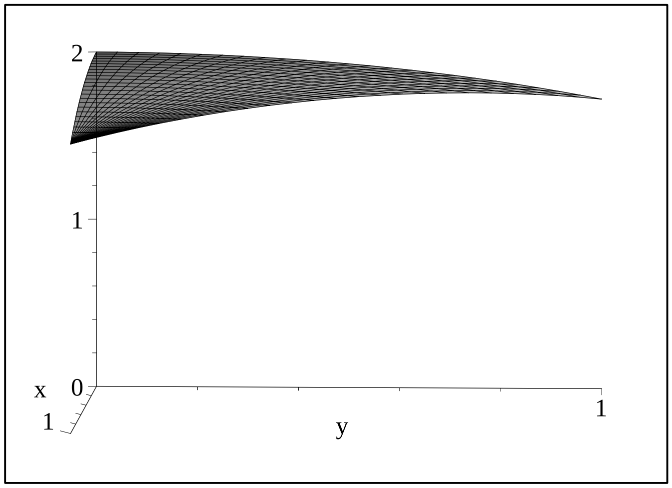
(b) Let $r(t) = \left(t^2 - t,\, 1+3t,\, t+t^3\right)$ for $0 \le t \le 2$ be a parametrization of the curve C. Compute $\int_C \vec{F} \cdot d\vec{r}$.

Since $\vec{F}$ is a conservative force field the value of the line integral depends only on the end points of the path. They are $r(2) = (2,7,10)$ and $r(0) = (0,1,0)$.

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \phi(2,7,10) - \phi(0,1,0) \\ &= 1798 - 0 = 1798\ . \end{aligned}$$

6. (15) Let S be that part of the sphere $x^2 + y^2 + z^2 = 4$, which lies above the triangular region in the x,y plane that is bounded by the lines $x + y = 1$, $x = 0$, and $y = 0$.

The figure is sketched below



(a) Find a parametric representation of S. Be sure to specify the domain of your parameters.

One parametric representation is

$$r(u, v) = \left(u, v, \sqrt{4 - u^2 - v^2}\right),$$

where $(u, v) \in D = \{(u, v) : 0 \le u \le 1,\ 0 \le v \le 1 - u\}$, which is a triangular domain in u,v space.

(b) Using the representation you gave in part a. set up an integral whose value is the area of S. You do not need to evaluate the integral.
We first need to find the partial derivatives of r with respect to u and v. They are

$$\begin{aligned}\frac{\partial r}{\partial u} &= \left(1, 0, \frac{-u}{\sqrt{4 - u^2 - v^2}}\right) \\ \frac{\partial r}{\partial v} &= \left(0, 1, \frac{-v}{\sqrt{4 - u^2 - v^2}}\right)\end{aligned}$$

The Jacobian of this transformation is $\left\|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right\|$, and this equals

$$\begin{aligned}\left\|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right\| &= \left\|\left(\frac{u}{\sqrt{4 - u^2 - v^2}}, \frac{v}{\sqrt{4 - u^2 - v^2}}, 1\right)\right\| \\ &= \sqrt{\frac{u^2}{4 - u^2 - v^2} + \frac{v^2}{4 - u^2 - v^2} + 1} = \sqrt{\frac{4}{4 - u^2 - v^2}}\end{aligned}$$

An integral whose value is the area of this surface is

$$\begin{aligned}\int\int_S dS &= \int\int_D \sqrt{\frac{4}{4 - u^2 - v^2}}\, du\, dv \\ &= \int_0^1 du \int_0^{1-u} \sqrt{\frac{4}{4 - u^2 - v^2}}\, dv \\ &\approx 0.523\end{aligned}$$

(c) Set up an integral whose value equals the length of that part of the boundary of S that lies above the line $x + y = 1$ in the x,y plane.

A parametric representation of this part of the boundary of S is given by

$$r(x) = \left(x, 1 - x, \sqrt{4 - x^2 - (1 - x)^2}\right) \text{ for } 0 \leq x \leq 1 \text{ .}$$

Thus, the length of this curve is

$$\begin{aligned}
\int_0^1 \|r'(x)\| \, dx &= \int_0^1 \left\| \left(1, -1, \frac{1 - 2x}{\sqrt{4 - x^2 - (1 - x)^2}}\right) \right\| \, dx \\
&= \int_0^1 \sqrt{1 + 1 + \frac{(1 - 2x)^2}{4 - x^2 - (1 - x)^2}} \, dx \\
&= \int_0^1 \sqrt{\frac{7}{3 + 2x - 2x^2}} \, dx \\
&\approx 1.45
\end{aligned}$$