1. (30) Define the following:

(a) $\lim_{(x,y)\to(-2,1)} f(x,y) = 5$

This means that for any $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < \|(x,y) - (-2,1)\| < \delta$, then $|f(x,y) - 5| < \epsilon$.

(b) the directional derivative of $f$ at the point $(2,3)$ in the direction $(1,1)$

Since the unit normal in the direction $(1,1)$ is $\left(1/\sqrt{2}, 1/\sqrt{2}\right)$, the directional derivative equals

$$\lim_{h\to 0} \frac{f\left(2 + h/\sqrt{2}, 3 + h/\sqrt{2}\right) - f(2,3)}{h}.$$

(c) $f(x,y)$ is differentiable at the point $(5,7)$

This means that there are numbers $\epsilon_1$ and $\epsilon_2$ such that

$$f(5 + \Delta x, 7 + \Delta y) = f(5,7) + \left.\frac{\partial f}{\partial x}\right|_{(5,7)} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{(5,7)} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where both $\epsilon_1$ and $\epsilon_2$ approach 0 as $(\Delta x, \Delta y) \to (0,0)$.

(d) the spherical variables $\rho$, $\theta$, and $\phi$

If $P$ is a point in $R^3$, then the spherical coordinates of $P$ represent

$\rho$ = distance of $P$ from the origin
= $\sqrt{x^2 + y^2 + z^2}$, where $(x,y,z)$ are the Cartesian coordinates of $P$

$\phi$ = angle that line from origin to $P$ makes with the positive $z$ axis, $0 \le \phi \le \pi$

$\theta$ = usual polar angle. That is, the point $\hat{P} = (x,y,0)$, which is the projection of $P$ onto the $x$,$y$ plane, has the polar coordinates $r$ and $\theta$, where $\theta$ is the angle the line from the origin to the point $\hat{P}$ makes with the positive $x$ axis, with $\theta$ lying between 0 and $2\pi$.

(e) the flux of a force field $F$ across a surface $S$

This is the surface integral of the scalar normal component of $\vec{F}$ over the surface $S$.

2. (25) Let $f(x,y,z) = 2xy - z + yz^2$.

   (a) Compute the directional derivative of $f$ at the point $(1,1,2)$ in the direction $\vec{N} = (5,0,-1)$.

   The gradient of $f$ equals $\nabla f = \left(2y,\, 2x + z^2,\, 2yz - 1\right)$, and its value at the point $(1,1,2)$ is $(2,6,3)$. The directional derivative equals

$$\begin{aligned} Df &= \nabla f|_{(1,1,2)} \cdot \frac{(5,0,-1)}{\sqrt{26}} \\ &= (2,6,3) \cdot \frac{(5,0,-1)}{\sqrt{26}} \\ &= \frac{7}{\sqrt{26}}. \end{aligned}$$

   (b) Find an equation for the tangent plane to the surface $f = 4$ at the point $(1,1,2)$.

$$(x-1, y-1, z-2) \cdot (2,6,3) = 0,$$

   or

$$2x + 6y + 3z = 14$$

   (c) What is the rate of change of $f$ at the point $(1,1,2)$ along any direction tangent to the plane of part b?

   Since any of these tangent directions is perpendicular to the gradient of $f$ at that point, all of these directional derivatives will equal 0.

3. (10) A force field, $\vec{F}$, is said to be conservative if there is a scalar valued function $\phi$ such that $\nabla\phi = \vec{F}$. Let $C$ denote any path with $P$ and $Q$ the beginning and terminal points of the path. Explain why the line integral of the tangential component of $F$ along the path $C$ depends only on $\phi$. That is, explain the formula

$$\int_C \vec{F} \cdot d\vec{r} = \phi(Q) - \phi(P).$$

Let $\Gamma(t) = (x(t), y(t), z(t))$ for $a \le t \le b$ be any parametrization of the curve $C$, such that $\Gamma(a) = P$ and $\Gamma(b) = Q$. Then we have

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_a^b (\phi_x(x(t), y(t), z(t)), \phi_y, \phi_z) \cdot \left(x', y', z'\right)\, dt \\ &= \int_a^b \left[\frac{d}{dt}\phi(\Gamma(t))\right]\, dt = \phi(\Gamma(b)) - \phi(\Gamma(a)) \\ &= \phi(Q) - \phi(P). \end{aligned}$$

4. (25) Let $S$ be the rectangular region in the $x$,$z$ plane that is bounded by the lines $x = 2$, $x = 0$, $z = 0$, and $z = 1$. Let $\vec{F} = (xy - z,\ x + 2z,\ \cos xy)$.

(a) curl$\vec{F}$ =

$$\begin{aligned} \text{curl}\vec{F} &= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ xy - z & x + 2z & \cos xy \end{bmatrix} \\ &= (-x \sin xy - 2, -1 + y \sin xy, 1 - x) \end{aligned}$$

(b) Compute the flux of curl$\vec{F}$ crossing the surface $S$ in the direction of increasing $y$ directly from the definition of flux as a surface integral.

$$\begin{aligned} \int\int_S \text{curl}\vec{F} \cdot dS &= \int\int_S (-2, -1, 1 - x) \cdot (0, 1, 0)\ \ dS \\ &= \int\int_S (-1)\ \ dS = -\text{area}\,(S) \\ &= -2 \end{aligned}$$

(c) Compute the flux of curl$\vec{F}$ crossing the surface $S$ in the direction of increasing $y$ by using Stoke's theorem.

Stoke's theorem says that $\int\int_S \text{curl}\vec{F} \cdot dS = \int_{\partial S} \vec{F} \cdot d\vec{r}$, where the path that is the boundary of $S$ is traced out in a direction compatable with the normal direction used in computing the surface integral. To use Stokes theorem in this problem we note that the normal direction is in the positive $y$ direction so if we look at $S$ from the positive $y$ axis the direction of integration around the boundary of $S$ must be in the counter clockwise direction.

Let $C_1$ represent that part of the boundary of $S$ for which $z = 0$, $C_2$ for $x = 0$, $C_3$ for $z = 1$, and $C_4$ for $x = 2$. Then we have

$$\begin{aligned} \int\int_S \text{curl}\vec{F} \cdot dS &= \int_{\partial S} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r} + \int_{C_4} \vec{F} \cdot d\vec{r} \\ &= \int_2^0 (0,\ x,\ 1) \cdot (dx, 0, 0) + \int_0^1 (-z,\ 2z,\ 1) \cdot (0, 0, dz) \\ &\quad + \int_0^2 (-1,\ x + 2,\ 1) \cdot (dx, 0, 0) + \int_1^0 (-z,\ 2 + 2z,\ 1) \cdot (0, 0, dz) \\ &= \int_2^0 0\ dx + \int_0^1 dz + \int_0^2 (-1)\, dx + \int_1^0 dz \\ &= -2 \end{aligned}$$

5. (20) Let $C$ denote the curve that is the intersection of the surfaces $x^2 + y^2 + z = 9$ and $z = 5$. Let $f(x, y, z) = x + y - 2z$.

(a) Find all critical points of $f$.

The critical points of $f$ are those points where the gradient of $f$ either does not exist or it equals $\vec{0}$. Since $f$ is a polynomial its gradient exists everywhere, so $(x, y, z)$ is a critical point of $f$ only if $\nabla f = \vec{0}$ at that point.

$$\nabla f = (1, 1, -2)$$

This can never be zero so $f$ has no critical points.

(b) Find the maximum value that the function $f$ attains on the curve $C$.

Use the Lagrange multiplier technique and find a solution to the equations

$$\begin{aligned} \nabla f + \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 &= (0, 0, 0) \\ g_1 &= x^2 + y^2 + z = 9 \\ g_2 &= z = 5\ . \end{aligned}$$

The first equation is equivalent to the following equations

$$\begin{aligned} 1 + 2\lambda_1 x &= 0 \\ 1 + 2\lambda_1 y &= 0 \\ -2 + \lambda_1 + \lambda_2 &= 0. \end{aligned}$$

Thus, we have $x = \dfrac{-1}{2\lambda_1} = y$. Since $x$, $y$, and $z$ must satisfy $g_1 = 0$, and $z = 5$, $\lambda_1$ satisfies

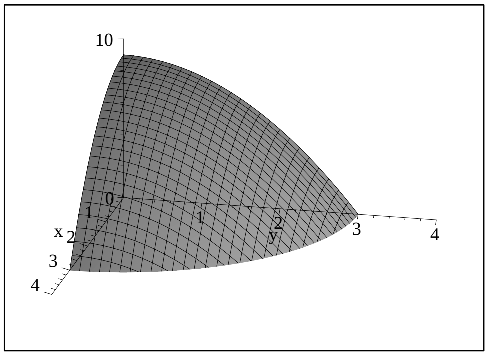
$$\begin{aligned} \frac{1}{4\lambda_1^2} + \frac{1}{4\lambda_1^2} &= 4 \\ \frac{1}{4\lambda_1^2} &= 2 \\ \frac{1}{2\lambda_1} &= \pm\sqrt{2} \end{aligned}$$

So we have $x = y = \pm\sqrt{2}$ Thus, the maximum value of $f$ on the curve $C$ occurs at $\left(\sqrt{2}, \sqrt{2}, 5\right)$ or at $\left(-\sqrt{2}, -\sqrt{2}, 5\right)$, and

$$\begin{aligned} f\left(\sqrt{2}, \sqrt{2}, 5\right) &= 2\sqrt{2} - 10 \\ f\left(-\sqrt{2}, -\sqrt{2}, 5\right) &= -2\sqrt{2} - 10 \end{aligned}$$

Hence the maximum value of $f$ on the curve $C$ is $2\sqrt{2} - 10$.

6. (40) Let $S$ denote the surface that encloses the region $E$, in $R^3$, which is bounded by $x = 0$, $y = 0$, $z = 0$, and $z = 9 - x^2 - y^2$. Let $S_1$, $S_2$, and $S_3$ denote those parts of $S$ that lie in the planes $x = 0$, $y = 0$, and $z = 0$ respectively, and let $S_4$ denote the remaining part of $S$. Let $\vec{F}(x, y, z) = (x, y, -z)$.
The region $E$ is shown below:




(a) Find the volume of $E$.

$$\begin{aligned}
\text{volume}(E) &= \iiint_E dV \\
&= \int_0^{\pi/2} d\theta \int_0^3 r\ dr \int_0^{9-r^2} dz \\
&= \frac{\pi}{2} \int_0^3 \left(9r - r^3\right)\ dr = \frac{\pi}{2} \left(\frac{9}{2}r^2 - \frac{r^4}{4}\right)\Big|_0^3 \\
&= \frac{81\pi}{8}
\end{aligned}$$

(b) Find the area of $S_4$.

The surface $S_4$ is the graph of the function $f(x, y) = 9 - x^2 - y^2$ for $x^2 + y^2 \leq 9$. Thus, the area of $S_4$ equals

$$\begin{aligned}
\text{area}(S_4) &= \iint_D \sqrt{1 + 4x^2 + 4y^2} dA \\
&= \int_0^{\pi/2} d\theta \int_0^3 r\sqrt{1 + 4r^2}\ dr \\
&= \frac{\pi}{2}\left(\frac{37\sqrt{37} - 1}{12}\right)
\end{aligned}$$

(c) Find the outward flux of $\vec{F}$ across each of the surfaces $S_1$, $S_2$, and $S_3$.

The flux of $\vec{F}$ across these surfaces equals

$$\begin{aligned}
\text{flux}\,(S_1\colon\ x=0) &= \int\!\!\int_{S_1} \vec{F}\cdot dS = \int\!\!\int_{S_1} (0,y,-z)\cdot(-1,0,0)\ dS = 0 \\
\text{flux}\,(S_2\colon\ y=0) &= \int\!\!\int_{S_2} \vec{F}\cdot dS = \int\!\!\int_{S_1} (x,0,-z)\cdot(0,-1,0)\ dS = 0 \\
\text{flux}\,(S_3\colon\ z=0) &= \int\!\!\int_{S_3} \vec{F}\cdot dS = \int\!\!\int_{S_1} (x,y,0)\cdot(0,0,-1)\ dS = 0
\end{aligned}$$

(d) Use the divergence theorem to find the outward flux of $\vec{F}$ across the surface $S$, and deduce what the outward flux of $\vec{F}$ across $S_4$ must equal.

The total outward flux across the surface $S$ equal

$$\begin{aligned}
\text{flux}\,(S) &= \int\!\!\int_{S} \vec{F}\cdot dS = \int\!\!\int\!\!\int_{E} \text{div}\left(\vec{F}\right)\ dV \\
&= \int\!\!\int\!\!\int_{E} dV = \text{volume}\,(E) = \frac{81\pi}{8}.
\end{aligned}$$

The outward flux across the surface $S_4$ plus the sums of the ourward fluxes across all of the other surfaces making up the surface $S$ must equal the total outward flux. Thus,

$$\begin{aligned}
\text{flux}\,(S_4) &= \text{flux}\,(S) - \text{flux}\,(S_1) - \text{flux}\,(S_2) - \text{flux}\,(S_3) \\
&= \text{flux}\,(S) \\
&= \frac{81\pi}{8}
\end{aligned}$$